On m-points uniformity tests on hyperspheres

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Abstract

When testing for the uniformity of directions, most classical approaches fall within the class of Sobolev tests. In this work, we propose generalizations of Sobolev tests via two new families of uniformity tests based on U- and V-statistics featuring kernels of arbitrary degree m that capture interactions among m-tuples of observations. Our tests encompass the classical Sobolev tests as a special case when $m = 2$. We demonstrate that the computation of these new V-statistics remains tractable even for large degrees and sample sizes, and we provide closed-form expressions for circular m-points test statistics. We investigate the asymptotic behavior of our m-points statistics under the null hypothesis and obtain non-standard results involving random Hermite polynomials. We also derive their asymptotic properties under both fixed and local alternatives. Through simulations, we show that tests with $m > 2$ yield important gains in power across several scenarios compared to classical Sobolev tests. Furthermore, we investigate the impact of m on the rotational invariance and the effect of invariantization in the asymptotic null distributions.

Keywords: Circular data; Spherical data; Uniformity tests.

1 Introduction

Directional statistics is concerned with data represented as directions, axes, or points on manifolds such as circles and spheres. It has broad applications across disciplines ranging from geology and meteorology to medicine and neuroscience. For a comprehensive overview of directional statistics, Mardia and Jupp (1999) offers a thorough introduction to classical methods, while Ley and Verdebout (2017) presents several modern methods. More recent developments and trends are reviewed in Pewsey and García-Portugués (2021).

Arguably, the problem of testing uniformity is fundamental in directional statistics and can be traced back to Bernoulli (1735), who explored the physical causes behind the inclinations of planetary orbits relative to the Sun's equatorial plane. This testing problem is straightforward to state: given a sample of random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n \sim \mathrm{P}$ on the hypersphere $\mathbb{S}^q := \{\mathbf{x} \in \mathbb{R}^{q+1} : \mathbf{x}'\mathbf{x} = 1\}$ of $\mathbb{R}^{q+1}$, with $q \geq 1$, the goal is to test $\mathcal{H}_0 : \mathrm{P} = \nu_q$ against $\mathcal{H}_1 : \mathrm{P} \neq \nu_q$, where ν_q denotes the uniform distribution on $\mathbb{S}^q$. This problem has been extensively studied in the literature. Notably, the elegant contributions of Beran (1968) and Giné (1975), who introduced 9. This class encompasses classical tests such as those by Rayleigh (1919), Bingham (1974), and Watson (1961). Recent Sobolev tests have focused on two main directions: improving detection power under multimodal alternatives, as in Pycke (2010) and Jammalamadaka et al (2020), and in Fernández-de-Marcos and García-Portugués (2023), who also addressed a second direction—data-driven tests—by proposing parameter-dependent tests whose parameters are chosen via cross-validation. In this second direction, Jupp (2008) introduced data-driven Sobolev tests with automatic truncation of the kernel.

Alongside Sobolev tests, another broad approach is projection-based tests. This approach reduces the problem to goodness-of-fit testing on $\mathbb{R}$ by projecting the sample. It includes the works of Cuesta-Albertos et al. (2009), proposing a Kolmogorov–Smirnov type test; García-Portugués et al. (2023),

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introducing a class of tests based on projection-integrated norms, extending the Watson (1961) test to $q > 1$ and proposing an Anderson–Darling test; and Borodavka and Ebner (2023), building another class of tests based on the powers of maximal projections.

Other recent contributions to testing uniformity have explored high-dimensional settings, such as Cutting et al. (2017), who studied local likelihood ratios, and Cai and Jiang (2012), who constructed a test based on the coherence of a random matrix; and novel methodologies, such as Hallin et al (2024), who employed optimal transport to build tests. See also the references therein. A review of uniformity testing is provided in García-Portugués and Verdebout (2018).

In the present work, we construct a class of *m-points tests* that generalizes Sobolev tests. As later seen, these tests achieve higher power than classical Sobolev tests against certain deviations from uniformity. To motivate the m-points tests construction, we begin with an overview of Sobolev tests. Sobolev tests reject the null hypothesis for large values of V-statistics defined as

$$S_{\phi}^{(n)} := \frac{1}{n} \sum_{i,j=1}^{n} \sum_{k=1}^{\infty} b_{q,k}(\phi)\, h_{q,k}(\mathbf{X}_i'\mathbf{X}_j), \text{ where } h_{q,k}(x) := \begin{cases} \cos\left(k \cos^{-1}(x)\right), & q = 1, \\ C_k^{(q-1)/2}(x), & q > 1, \end{cases} \tag{1}$$

and C_p^{λ} denotes the pth Gegenbauer polynomial of order λ. Gegenbauer polynomials C_p^{λ} are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $t \mapsto (1 - t^2)^{\lambda - 1/2}$. Therefore, the coefficients $b_{q,k}(\phi)$ in (1) can be seen as the projections of a kernel $\phi : [-1, 1] \to \mathbb{R}$ onto the orthonormal basis $\{h_{q,k}\}_{k=1}^{\infty}$, that is,

$$b_{q,k}(\phi) := \frac{1}{c_{q,k}} \int_{-1}^{1} \phi(t)\, h_{q,k}(t)\, (1 - t^2)^{(q-2)/2}\, \mathrm{d}t, \tag{2}$$

where the constant $c_{q,k}$ is defined as $c_{q,k} := \frac{\omega_q}{\omega_{q-1}} a_{q,k}^{-1} C_k^{(q-1)/2}(1)$ for $q > 1$, and $c_{1,k} := a_{1,k}^{-1}\,(1 + \delta_{k0})\,\pi$ for $q = 1$, where $\delta_{k\ell}$ is the usual Kronecker delta. Before, we used

$$a_{q,k} := 2 \cdot \mathbb{1}_{\{q=1\}} + (1 + 2k/(q-1)) \cdot \mathbb{1}_{\{q>1\}},$$

with $\mathbb{1}$ denoting the indicator function.

Some assumptions on the kernel ϕ seem natural:

(*c1*) $\phi \in L_q^2[-1, 1]$, where $L_q^2[-1, 1]$ is the space of square integrable functions with respect to the weight $t \mapsto (1 - t^2)^{(q-2)/2}$.

(*c2*) $\sum_{k=1}^{\infty} |w_k|\, d_{q,k} < \infty$, where $w_k := a_{q,k}^{-1} b_{k,q}(\phi)$ and $d_{q,k} := \binom{q+k}{k} - \binom{q+k-2}{k-2}$.

Condition (*c1*) ensures a finite variance for $S_{\phi}^{(n)}$ under $\mathcal{H}_0$, while condition (*c2*) guarantees that the statistic $S_{\phi}^{(n)}$ is well-defined. For $q > 1$, this follows from

$$\sum_{k=1}^{\infty} \left| b_{q,k}(\phi)\, C_k^{(q-1)/2}(t) \right| \le \sum_{k=1}^{\infty} |w_k|\, a_{q,k}\, C_k^{(q-1)/2}(1) = \sum_{k=1}^{\infty} |w_k|\, d_{q,k} < \infty,$$

as condition (*c2*) guarantees the (uniform on $[-1, 1]$) convergence of $\sum_{k=1}^{\infty} b_{q,k}(\phi)\, h_{q,k}(t)$. While condition (*c2*) is necessary for V-statistics, it is not required for U-statistics, which exclude diagonal terms ($i = j$) and thus permit "extreme" kernels that do not satisfy (*c2*), as the test statistic introduced in Fernández-de-Marcos and García-Portugués (2024) demonstrates.

From (1), it is clear that the kernel ϕ characterizes the behavior of $S_{\phi}^{(n)}$ by weighting the pairwise distance between observations, through the $\mathbf{X}_i'\mathbf{X}_j$'s. Typically, ϕ is chosen to emphasize proximity, with higher values as $\mathbf{X}_i'\mathbf{X}_j \to 1$, since clustering structures are considered as strong evidence against uniformity. More generally, ϕ can be seen as the restriction of a symmetric function $\bar{\phi} : \mathbb{S}^q \times \mathbb{S}^q \to \mathbb{R}$ with $\bar{\phi}(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x}'\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^q$. This broader perspective motivates extending Sobolev tests beyond pairwise interactions, leading to the development of m-points tests. These are based on kernels $h : (\mathbb{S}^q)^m \to \mathbb{R}$ of degree m, that allow the statistic to capture higher-order interactions among m-tuples of observations. Such kernels are constructed by expanding square-integrable functions on $(\mathbb{S}^q)^m$ in an orthonormal basis of the corresponding Hilbert space. The test statistics are thus determined by the sequence of basis coefficients, which we will refer to as *weights*.

The contributions of this paper are manifold. First, we introduce two types of m-points tests, respectively based on U- and V-statistics. Both tests are seen to be equivalent to Sobolev tests in the specific case where $m = 2$. We also provide closed-form expressions for m-points kernels on $\mathbb{S}^1$ using weights from classical Sobolev tests. Second, we obtain the asymptotic null distribution of our U- and V-statistics, first for *finite m-points tests* with truncated kernels, and then for *infinite m-points tests*; this distinction is emphasized due to the practical relevance of finite tests when closed-form expressions are unavailable. The results obtained are non-standard. In particular, some weak limits under the null hypothesis involve random Hermite polynomials—an unusual feature in classical statistical frameworks. These polynomials emerge naturally in the limiting behavior of the test statistics. Third, we derive the asymptotic behavior under fixed and local alternatives. These asymptotic results on U- and V-statistics drive the definition of critical regions for m-points tests, which depend on both the type of statistic (U or V) and the parity of m. Fourth, we discuss the rotational invariance of our tests. This property holds for $m = 2$, but generally fails for $m > 2$. To recover rotational invariance, two approaches are proposed: (i) a quasi-invariant test via random rotations and aggregation of p-values that is practically feasible in any dimension, and (ii) an invariant V-statistic based on a kernel whose closed-form expression and asymptotic distribution are derived for the circular case.

We study the new tests through a series of numerical experiments with four main objectives. First, we compare the performance of tests based on the proposed U- and V-statistics (which we refer to as m-points U- and V-tests, respectively), observing that the latter generally achieve higher power under local alternatives and also offer a substantial computational advantage when finite kernels are used, since the complexity is reduced to $O(n)$. This efficiency gain comes from a kernel reformulation that avoids a complete computation, as required by U-statistics, and motivates our focus on V-statistics in the remainder of the study. Second, we explore the empirical behavior of the proposed tests under the null hypothesis, observing that introducing a correction factor in the U-statistics improves the behavior of the corresponding tests with small sample sizes. Third, we demonstrate that the V-tests attain increased power over the Sobolev counterparts under various fixed alternative scenarios. In particular, $(m > 2)$-points V-tests show better detection power under multimodal alternatives. Finally, we assess the performance of the rotation-invariant versions: quasi-rotation-invariant V-tests still show higher power compared to the $m = 2$ case, and for the rotation-invariant V-statistics, we illustrate that, in low dimensions, the invariant asymptotic distribution closely matches that of the original m-points statistics, particularly in the upper tail.

The rest of the paper is organized as follows. Section 2 presents the required fundamentals of spherical harmonics. Section 3 introduces the m-points class of test statistics, including closed-form expressions for the circular case. Section 4 derives the asymptotic distribution of the statistics under the null hypothesis, while Section 5 explores the consistency against fixed alternatives and extends the asymptotic results to local alternatives. Section 6 defines the tests based on the m-points statistics, following a discussion on the corresponding critical regions. In Section 7, the rotational invariance properties of m-points tests are studied. Finally, Section 8 presents comprehensive numerical experiments showing the validity of the theoretical results and the advantages of the new class of tests. All proofs and additional simulations are provided in the Supplementary Materials (SM).

2 Spherical harmonics

Let $L^2(\mathbb{S}^q, \nu_q)$ denote the space of square-integrable functions defined on $\mathbb{S}^q$ with respect to the uniform distribution, ν_q, and define the inner product $\langle f, g\rangle := \int_{\mathbb{S}^q} f(\mathbf{x})\, g(\mathbf{x})\, \mathrm{d}\nu_q(\mathbf{x})$ for any $f, g \in L^2(\mathbb{S}^q, \nu_q)$. $L^2(\mathbb{S}^q, \nu_q)$ is a separable Hilbert space, hence, an orthonormal basis can be obtained. We work with a specific basis related to the spherical harmonics.

The space of real homogeneous polynomials of degree $k \geq 1$ and $q+1$ variables is denoted by $\mathcal{P}_k^{q+1}$. Let $\mathcal{H}_k^{q+1}$ be the subspace of $\mathcal{P}_k^{q+1}$ of real harmonic polynomials, that is, $\mathcal{H}_k^{q+1} := \{P \in \mathcal{P}_k^{q+1} : \Delta P = 0\}$, where Δ denotes the Laplacian operator $\Delta := \partial_1^2 + \cdots + \partial_{q+1}^2$. Spherical harmonics of degree k are the restrictions of real harmonic polynomials to $\mathbb{S}^q$. We also denote by $\mathcal{H}_k^{q+1}$ the space of spherical harmonics of degree k. The dimension of the space of spherical harmonics has the well-known expression given by $\dim \mathcal{H}_k^{q+1} = d_{q,k}$, see, e.g., Corollary 1.1.4. in Dai and Xu (2013). Spherical harmonics of different degrees $k \neq \ell$, $\psi \in \mathcal{H}_k^{q+1}$, $\phi \in \mathcal{H}_\ell^{q+1}$ are orthogonal $\langle \psi, \phi\rangle = 0$, thus combining orthonormal basis of $\mathcal{H}_k^{q+1}$ for different degrees results in an orthonormal basis of their direct sum. In addition, the collection of spherical harmonics is dense in $L^2(\mathbb{S}^q, \nu_q)$, hence $L^2(\mathbb{S}^q, \nu_q)$ is equal to the direct sum of $\mathcal{H}_k^{q+1}$, for $k = 0, 1, \ldots$, see, e.g., Theorem 2.2.2 in Dai and Xu (2013). In the following, we define a specific basis of spherical harmonics, which can be found in Theorem 1.5.1 in Dai and Xu (2013) with appropriate corrections to account for errata.

For $k = 0$, regardless of the dimension $q \geq 1$, $d_{q,0} = 1$ and an orthonormal basis of $\mathcal{H}_0^{q+1}$ is $\{g_{0,1}\}$ with $g_{0,1}(\mathbf{x}) := 1$ for all $\mathbf{x} \in \mathbb{S}^q$. For $q = 1$, let $k \geq 1$. Since $d_{1,k} = 2$, using polar coordinates $\mathbf{x} = (\cos\theta, \sin\theta)$, $0 \leq \theta < 2\pi$, the functions $g_{k,1}(\mathbf{x}) := \sqrt{2}\cos(k\theta)$ and $g_{k,2}(\mathbf{x}) := \sqrt{2}\sin(k\theta)$, for every $\mathbf{x} \in \mathbb{S}^1$, form an orthonormal basis of $\mathcal{H}_k^{q+1}$.

For $q \geq 2$, let $k \geq 1$, and consider the hyperspherical coordinates

$$\begin{cases} x_1 = \sin\theta_q \cdots \sin\theta_2 \sin\theta_1, \\ x_2 = \sin\theta_q \cdots \sin\theta_2 \cos\theta_1, \\ \quad\vdots \\ x_q = \sin\theta_q \cos\theta_{q-1}, \\ x_{q+1} = \cos\theta_q, \end{cases} \tag{3}$$

with $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_j \leq \pi$ for $j = 2, 3, \ldots, q$, where $\mathbf{x} = (x_1, \ldots, x_q, x_{q+1})' \in \mathbb{S}^q$. Let $\mathcal{M}_k = \{\mathbf{m} \in \mathbb{N}_0^{q+1} : |\mathbf{m}| := m_1 + \cdots + m_{q+1} = k \text{ and } m_{q+1} \in \{0, 1\}\}$. For $\mathbf{m} \in \mathcal{M}_k$, let

$$\zeta_{\mathbf{m}}(s) := \begin{cases} \cos(m_q s) & \text{if } m_{q+1} = 0, \\ \sin((m_q+1)s) & \text{if } m_{q+1} = 1 \end{cases} \quad \text{and } B_{\mathbf{m}} := b_{\mathbf{m}} \prod_{j=1}^{q-1} \frac{m_j! (\frac{q-j+2}{2})_{|\mathbf{m}^{j+1}|} (m_j + \lambda_j)}{(2\lambda_j)_{m_j} (\frac{q-j+1}{2})_{|\mathbf{m}^{j+1}|} \lambda_j},$$

where we write $|\mathbf{m}^j| = m_j + \cdots + m_{q+1}$ and $\lambda_j := |\mathbf{m}^{j+1}| + (q-j)/2$ for any $j = 1, 2, \ldots, q-1$, and where $b_{\mathbf{m}} := 2$ if $m_q + m_{q+1} > 0$ and $b_{\mathbf{m}} := 1$ otherwise. Then, we define $\varphi_{\mathbf{m}} : \mathbb{S}^q \to \mathbb{R}$ in spherical coordinates by

$$\varphi_{\mathbf{m}}(\mathbf{x}) := \sqrt{B_{\mathbf{m}}}\, \zeta_{\mathbf{m}}(\theta_1) \prod_{j=1}^{q-1} (\sin\theta_{q-j+1})^{|\mathbf{m}^{j+1}|} C_{m_j}^{\lambda_j}(\cos\theta_{q-j+1}),$$

the collection $\{\varphi_{\mathbf{m}} : \mathbf{m} \in \mathcal{M}_k\}$ is a real orthonormal basis of $\mathcal{H}_k^{q+1}$, in the sense that $\langle \varphi_{\mathbf{m}}, \varphi_{\tilde{\mathbf{m}}}\rangle = \delta_{\mathbf{m}\tilde{\mathbf{m}}}$ for any $\mathbf{m}, \tilde{\mathbf{m}} \in \mathcal{M}_k$. Note that this orthonormal basis is indeed of cardinality $d_{q,k}$.

Let us enumerate the $\mathbf{m}$'s in $\mathcal{M}_k$ as $\mathbf{m}_r$, $r = 1, 2, \ldots, d_{q,k}$, in "colex" (colexicographic) order, that is, given $\mathbf{m}, \tilde{\mathbf{m}} \in \mathcal{M}_k$, $\mathbf{m} <^{\mathrm{c}} \tilde{\mathbf{m}}$ if and only if there exists an $i > 0$ such that for all $j > i$, $(m_j = \tilde{m}_j) \wedge (m_i < \tilde{m}_i)$. Finally, we define $g_{k,r}(\mathbf{x}) := \varphi_{\mathbf{m}_r}(\mathbf{x})$.

3 m-points test statistics

3.1 Motivation and definition

Let $m \geq 2$ be an integer. Consider the space $L^2((\mathbb{S}^q)^m, (\nu_q)^m) \equiv L^2((\mathbb{S}^q)^m)$ of functions defined on $(\mathbb{S}^q)^m$ that are square integrable with respect to $(\nu_q)^m$. Together with the inner product

$$\langle f, g\rangle_m := \int_{(\mathbb{S}^q)^m} f(\mathbf{x}_1, \ldots, \mathbf{x}_m) g(\mathbf{x}_1, \ldots, \mathbf{x}_m) \,\mathrm{d}(\nu_q)^m(\mathbf{x}_1, \ldots, \mathbf{x}_m),$$

for $f, g \in L^2((\mathbb{S}^q)^m)$, $L^2((\mathbb{S}^q)^m)$ is a separable Hilbert space, in which the set of functions of the form $(\mathbf{x}_1, \ldots, \mathbf{x}_m) \mapsto g_{k_1,r_1}(\mathbf{x}_1) \cdots g_{k_m,r_m}(\mathbf{x}_m)$ is an orthonormal basis of the space. Thus, every function $h \in L^2((\mathbb{S}^q)^m)$ can be expressed as

$$h(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{k_1,\ldots,k_m=1}^{\infty} \sum_{r_1=1}^{d_{q,k_1}} \cdots \sum_{r_m=1}^{d_{q,k_m}} \langle h, g_{k_1,r_1} \cdots g_{k_m,r_m}\rangle_m \, g_{k_1,r_1}(\mathbf{x}_1) \cdots g_{k_m,r_m}(\mathbf{x}_m), \tag{4}$$

with the series converging in mean square and $\sum_{k_1,\ldots,k_m=1}^{\infty} \sum_{\mathbf{r}} \langle h, g_{k_1,r_1} \ldots g_{k_m,r_m}\rangle_m^2 < \infty$. To ease the notation, we write $\sum_{\mathbf{r}}$ for $\sum_{r_1=1}^{d_{q,k_1}} \cdots \sum_{r_m=1}^{d_{q,k_m}}$ in the sequel. Conversely, the Riesz–Fischer Theorem ensures that given a sequence of weights $\{w_{k_1,\ldots,k_m,r_1,\ldots,r_m}\}$ that is square-summable

$$\sum_{k_1,\ldots,k_m=1}^{\infty} \sum_{\mathbf{r}} w_{k_1,\ldots,k_m,r_1,\ldots,r_m}^2 < \infty, \tag{5}$$

then there is a function $h_w \in L^2((\mathbb{S}^q)^m)$ such that $\langle h_w, g_{k_1,r_1} \cdots g_{k_m,r_m}\rangle_m = w_{k_1,\ldots,k_m,r_1,\ldots,r_m}$. The m-points class of test statistics leverages this characterization: rather than specifying a kernel directly, we construct it via its spherical harmonic expansion using a chosen sequence of weights $\{w_{k_1,\ldots,k_m,r_1,\ldots,r_m}\}$ that satisfies (5).

The flexibility offered by kernels of degree m is substantial. However, selecting an appropriate sequence of weights to ensure desirable characteristics on the statistic is challenging. For this reason, we restrict our attention to simplified kernels defined on $(\mathbb{S}^q)^m$, inspired by the structure of Sobolev kernels, which depend only on the inner product of pairwise observations, $\mathbf{X}_i'\mathbf{X}_j$, and whose coefficients exhibit certain simplifications as a result. To see this, fix $m = 2$ and consider $h \in L^2((\mathbb{S}^q)^2)$ such that $h(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x}'\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^q$, for some function $\phi : [-1, 1] \to \mathbb{R}$. This assumption implies that h is not only symmetric on its arguments but also invariant under arbitrary rotations of the sample. Then, let

$$\begin{aligned} w_{k_1,k_2,r_1,r_2} &:= \langle h, g_{k_1,r_1} g_{k_2,r_2}\rangle_2 \\ &= \frac{1}{\omega_q} \int_{\mathbb{S}^q} \left(\int_{\mathbb{S}^q} \phi(\mathbf{x}_1'\mathbf{x}_2) g_{k_1,r_1}(\mathbf{x}_1) \,\mathrm{d}\sigma_q(\mathbf{x}_1) \right) g_{k_2,r_2}(\mathbf{x}_2) \,\mathrm{d}\nu_q(\mathbf{x}_2) \\ &= \frac{w_{k_1}(\phi)}{\omega_q} \int_{\mathbb{S}^q} g_{k_1,r_1}(\mathbf{x}_2) g_{k_2,r_2}(\mathbf{x}_2) \,\mathrm{d}\nu_q(\mathbf{x}_2) = \frac{w_{k_1}(\phi)}{\omega_q} \delta_{k_1 k_2} \delta_{r_1 r_2}, \end{aligned} \tag{6}$$

where we applied the Funk–Hecke Theorem, see, e.g., Theorem 1.2.9 in Dai and Xu (2013), σ_q denotes the Lebesgue measure on $\mathbb{S}^q$, $\omega_q := \sigma_q(\mathbb{S}^q) = 2\pi^{(q+1)/2}/\Gamma((q + 1)/2)$ is the surface area of $\mathbb{S}^q$, and

$$w_{k_1}(\phi) := \frac{\omega_{q-1}}{h_{q,k_1}(1)} \int_{-1}^{1} \phi(t)\, h_{q,k_1}(t)\, (1 - t^2)^{q/2-1} \,\mathrm{d}t = \frac{c_{q,k_1}\, \omega_{q-1}}{h_{q,k_1}(1)}\, b_{q,k_1}(\phi).$$

Therefore, (6) shows how the rotational invariance inherent in ϕ induces a simplified structure on $\{w_{k_1,k_2,r_1,r_2}\}$. This principle can be extended to kernels of degree m by considering weights $\{w_{\mathbf{k},\mathbf{r}}\}$ with $\mathbf{k} = (k_1, \ldots, k_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$, satisfying the following:

($w1$) *Diagonality*: $w_{\mathbf{k},\mathbf{r}} = 0$ if there is some $i \neq j$ such that $k_i \neq k_j$ or $r_i \neq r_j$, with $i, j \in \{1, \ldots, m\}$.

($w2$) *Homogeneity*: if $\mathbf{k} = (k, \ldots, k)$, $k \geq 1$, then $w_{\mathbf{k},\mathbf{r}} = w_{\mathbf{k},\mathbf{r}^*}$ for any $\mathbf{r} = (r, \ldots, r)$ and $\mathbf{r}^* = (r^*, \ldots, r^*)$, with $r, r^* \in \{1, \ldots, d_{q,k}\}$.

Notably, the general sequence of coefficients $\{w_{\mathbf{k},\mathbf{r}}\}$ is reduced to a sequence $\{w_k\}_{k=1}^{\infty}$. This allows the direct use of classical sequences of Sobolev weights to build kernels of degree m. Based on these simplified kernels of degree m, we define the class of m-points test statistics.

Definition 3.1 (m-points test statistic)**.** *Let $\{\mathbf{X}_i\}_{i=1}^n$ be a sample on $\mathbb{S}^q$, $m \geq 2$, and $w := \{w_k\}_{k=1}^{\infty}$ be a real sequence such that $\sum_{k=1}^{\infty} w_k^2 d_{q,k} < \infty$. The* m-points test statistic *associated with w is defined as*

$$V_{m,w}^{(n)} := n^{-m/2} \sum_{i_1,\ldots,i_m=1}^{n} \Phi_w(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m})$$

in its V-form, and as

$$U_{m,w}^{(n)} := n^{m/2} \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \Phi_w(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m})$$

in its U-form, with the kernel induced by w given by

$$\Phi_w(\mathbf{X}_1, \ldots, \mathbf{X}_m) := \sum_{k=1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \psi_{k,r}(\mathbf{X}_1, \ldots, \mathbf{X}_m) := \sum_{k=1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \prod_{j=1}^{m} g_{k,r}(\mathbf{X}_j). \tag{7}$$

Note that the square summability condition in (5) simplifies to $\sum_{k=1}^{\infty} w_k^2 d_{q,k} < \infty$ due to the structure of diagonal, homogeneous weights.

Motivated by the use of simplified weights, the relation between Sobolev and m-points tests becomes evident: Sobolev statistics are equivalent to 2-points statistics. Let $\phi : [-1, 1] \to \mathbb{R}$ be a function satisfying conditions ($c1$) and ($c2$), and let $b_{q,k}(\phi)$ be the projections of ϕ as given in (2). Let the 2-points V-statistic, $V_{2,w}^{(n)}$, induced by the sequence of weights $w_k = a_{q,k}^{-1} b_{q,k}(\phi)$. Then,

$$V_{2,w}^{(n)} = \frac{1}{n} \sum_{i,j=1}^{n} \sum_{k=1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} g_{k,r}(\mathbf{X}_i) g_{k,r}(\mathbf{X}_j) = \frac{1}{n} \sum_{i,j=1}^{n} \sum_{k=1}^{\infty} b_{q,k}(\phi) h_{q,k}(\mathbf{X}_i' \mathbf{X}_j) = S_{\phi}^{(n)},$$

where $h_{q,k}$ is given in (1) and we used the addition formula of spherical harmonics

$$\sum_{r=1}^{d_{q,k}} g_{k,r}(\mathbf{x}) g_{k,r}(\mathbf{y}) = a_{q,k} h_{q,k}(\mathbf{x}'\mathbf{y}) \tag{8}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^q$. For the 2-points U-statistic, $U_{2,w}^{(n)}$, the equivalence holds up to an affine transformation,

$$U_{2,w}^{(n)} = \frac{n}{n-1} \left(S_{\phi}^{(n)} - \phi(1) \right),$$

where $\phi(1) = \sum_{k=1}^{\infty} w_k \, d_{q,k} < \infty$.

3.2 Closed-form expressions

Closed-form expressions are useful for avoiding truncation in the computation of statistics with infinite kernel expansions. This section derives such closed-form expressions for m-points test statistics in the circular case ($q = 1$) induced by several classical Sobolev weights. We work in polar coordinates, letting $\mathbf{x} = (\cos\theta, \sin\theta)' \in \mathbb{S}^1$, with $0 \leq \theta < 2\pi$ denoting the polar angle.

Proposition 3.1. *Let $m \geq 2$ be an even integer, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ be the polar angles of $(\mathbf{X}_1, \ldots, \mathbf{X}_m)'$, and w be a real sequence such that $\sum_{k=1}^{\infty} w_k^2 d_{q,k} < \infty$. Let $\Phi_{m,w}(\boldsymbol{\theta}) := \Phi_w(\mathbf{X}_1, \ldots, \mathbf{X}_m)$. Then,*

$$\Phi_{m,w}(\boldsymbol{\theta}) = \sum_{k=1}^{\infty} \sum_{\mathbf{e}} w_k \nu_{\mathbf{e}} \cos\left(k\mathbf{e}'\boldsymbol{\theta}\right), \tag{9}$$

where the sum $\sum_{\mathbf{e}}$ is carried out over $\mathbf{e} := (e_1, \ldots, e_m)' \in \{1\} \times \{-1, 1\}^{m-1}$, $p_{\mathbf{e}} := \prod_{i=1}^{m} e_i$, and $\nu_{\mathbf{e}} := 2^{1-m/2}\left(1 + (-1)^{m/2} p_{\mathbf{e}}\right)$.

When the series $\sum_{k=1}^{\infty} w_k \cos\left(k\mathbf{e}'\boldsymbol{\theta}\right)$ converges, the sum and the series in (9) can be interchanged, resulting in a finite sum that can be related to the classical Sobolev kernels. Indeed, whenever a closed-form expression is available for the Sobolev kernel $\psi(\theta) := \sum_{k=1}^{\infty} w_k \cos(k\theta)$, for $\theta \in (0, \pi]$, a related closed-form expression $\bar{\psi}$ can be extended to $\theta \in \mathbb{R}$ by periodicity $\bar{\psi}(|\theta|) := \psi\left(2\pi\left\{|\theta|/(2\pi)\right\}\right)$ and exploiting symmetry $\bar{\psi}(-|\theta|) := \bar{\psi}(|\theta|)$, where $\{x\} := x - \lfloor x \rfloor$. The next result collects a set of classical Sobolev weights for which such closed-form expressions are derived.

Corollary 3.1. *Let $m \geq 2$ be an even integer. Under the notation of Proposition 3.1, let $\tau_{\mathbf{e}} := \nu_{\mathbf{e}}/2$ and $\tilde{\theta}_{\mathbf{e}} := \{\mathbf{e}'\boldsymbol{\theta}/2\pi\}$. Then, denoting $\Phi_{m,w}(\boldsymbol{\theta}) := \Phi_w(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ for a general weight sequence w, we have:*

(i) (m-Watson) $w_{\mathrm{W}} = \{(\sqrt{2}\,\pi k)^{-2}\}_{k=1}^{\infty}$,

$$\Phi_{m,w_{\mathrm{W}}}(\boldsymbol{\theta}) = \sum_{\mathbf{e}} \tau_{\mathbf{e}} B_2(\tilde{\theta}_{\mathbf{e}}),$$

where B_k is the kth degree Bernoulli polynomial.

(ii) (m-Anderson–Darling) $w_{\mathrm{AD}} = \left\{(\sqrt{2\pi} k)^{-2} \int_0^{\pi} \frac{1-\cos(2k\theta)}{(\pi-\theta)\theta}\,\mathrm{d}\theta\right\}_{k=1}^{\infty}$ *and* $\tilde{\theta}_{\mathbf{e}} \neq 0$,

$$\Phi_{m,w_{\mathrm{AD}}}(\boldsymbol{\theta}) = \sum_{\mathbf{e}} \tau_{\mathbf{e}}\left[1 + 2\left[|\tilde{\theta}_{\mathbf{e}}| \log|\tilde{\theta}_{\mathbf{e}}| + (1 - |\tilde{\theta}_{\mathbf{e}}|)\log(1 - |\tilde{\theta}_{\mathbf{e}}|)\right]\right].$$

(iii) (m-Rothman) $w_{\mathrm{R}} = \{\sin^2(k\pi t_\wedge)(\pi k)^{-2}\}_{k=1}^{\infty}$, *where* $t_\wedge := \min(t, 1-t)$ *for* $t \in (0, 1)$,

$$\Phi_{m,w_{\mathrm{R}}}(\boldsymbol{\theta}; t) = \sum_{\mathbf{e}} \tau_{\mathbf{e}}\left[\left(t_\wedge - \left\{\frac{|\mathbf{e}'\boldsymbol{\theta}|}{2\pi}\right\}\right)_+ + \left(t_\wedge - 1 + \left\{\frac{|\mathbf{e}'\boldsymbol{\theta}|}{2\pi}\right\}\right)_+ + t_\wedge^2\right].$$

(iv) (m-Pycke) $w_{\mathrm{P}} = \{(2k)^{-1}\}_{k=1}^{\infty}$,

$$\Phi_{m,w_{\mathrm{P}}}(\boldsymbol{\theta}) = \sum_{\mathbf{e}} -\tau_{\mathbf{e}} \log\left(2\sin(\pi\tilde{\theta}_{\mathbf{e}})\right).$$

(v) (m-Smooth maximum) $w_{\mathrm{S}} = \{e^{-\kappa}\mathcal{I}_k(\kappa)\}_{k=1}^{\infty}$ *for* $\kappa > 0$,

$$\Phi_{m,w_{\mathrm{S}}}(\boldsymbol{\theta}; \kappa) = \sum_{\mathbf{e}} \tau_{\mathbf{e}}\left(e^{\kappa(\cos(\mathbf{e}'\boldsymbol{\theta})-1)} - e^{-\kappa}\mathcal{I}_0(\kappa)\right).$$

(vi) (m-Poisson) $w_{\mathrm{Pois}} = \{\rho^k\}_{k=1}^{\infty}$ *for* $\rho \in (0, 1)$,

$$\Phi_{m,w_{\mathrm{Pois}}}(\boldsymbol{\theta}; \rho) = \sum_{\mathbf{e}} \tau_{\mathbf{e}}\left(\frac{1-\rho^2}{1 - 2\rho\cos(\mathbf{e}'\boldsymbol{\theta}) + \rho^2} - 1\right).$$

4 Null asymptotics

In this section, we derive the asymptotic distribution of the class of m-points test statistics introduced in Definition 3.1 under the null hypothesis, assuming the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n \sim \nu_q$ is independent and identically distributed (iid). The analysis begins with the limiting distribution of finite m-points kernels and is then extended to the infinite kernel case. Throughout this and the remaining sections, we use the notation $\{Z_{k,r} : k \geq 1, 1 \leq r \leq d_{q,k}\}$ for a collection of independent standard normal random variables, H_m for the mth order Hermite polynomial, given recursively by $H_0(x) := 1$ and $H_{m+1}(x) := x\, H_m(x) - H_m'(x)$, and $\rightsquigarrow$ for weak convergence.

4.1 Finite m-points statistics

We define a K-finite m-points kernel as the truncated version of the kernel given in (7),

$$\Phi_{w,K}(\mathbf{X}_1, \ldots, \mathbf{X}_m) := \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k\, \psi_{k,r}(\mathbf{X}_1, \ldots, \mathbf{X}_m).$$

Equivalently, $\Phi_{w,K}$ can be seen as an m-points kernel induced by weights w such that $w_k = 0$ for all $k > K$, thus the square summability condition of w is trivially satisfied. We use the terminology infinite m-points kernel to refer to the m-points kernel (7), i.e., $K = \infty$. We denote by $U^{(n)}_{m,w,K}$ and $V^{(n)}_{m,w,K}$ the K-finite m-points U- and V-statistics, respectively. The next proposition provides the null asymptotic distribution of finite U- and V-statistics. This result underpins the construction of asymptotic tests based on these statistics, which is the focus of Section 6.

Proposition 4.1. *Let $q \geq 1$, $K \geq 1$, $m \geq 2$, and w be a real sequence. Then, under $\mathcal{H}_0$ and as $n \to \infty$:*

(i) $U^{(n)}_{m,w,K} \rightsquigarrow U^{\infty}_{m,w,K} := \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k H_m(Z_{k,r})$;

(ii) $V^{(n)}_{m,w,K} \rightsquigarrow V^{\infty}_{m,w,K} := \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k Z^m_{k,r}$.

4.2 Infinite m-points statistics

Proposition 4.1 can be extended to the case of infinite m-points statistics, provided that certain natural conditions on the sequence of weights are satisfied. For U-statistics, the square summability of the weights w suffices, as it guarantees the finiteness of the variance under $\mathcal{H}_0$. We now state the result for U-statistics which, similarly to Proposition 4.1, is crucial for constructing tests based on $U^{(n)}_{m,w}$ (Section 6).

Theorem 4.1. *Let $q \geq 1$, $m \geq 2$, and w be a real sequence such that $\sum_{k=1}^{\infty} w_k^2\, d_{q,k} < \infty$. Then, under $\mathcal{H}_0$ and as $n \to \infty$,*

$$U^{(n)}_{m,w} \rightsquigarrow U^{\infty}_{m,w} := \sum_{k=1}^{\infty} \sum_{r=1}^{d_{q,k}} w_k H_m(Z_{k,r}). \tag{10}$$

A corrected version of the U-statistic,

$$U^{*(n)}_{m,w} := A^{-1}_{n,m}\, U^{(n)}_{m,w} = n^{-m/2}\, m! \sum_{1 \leq i_1 < \cdots < i_m \leq n} \Phi_w(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}),$$

where $A_{n,m} := n^m (n-m)!/n!$, has the same asymptotic distribution as in (10). However, the test based on $U^{*(n)}_{m,w}$ presents an improved finite-sample accuracy: when using asymptotic critical values derived from Theorem 4.1, $U^{*(n)}_{m,w}$ yields more accurate results for small sample sizes compared to the uncorrected statistic $U^{(n)}_{m,w}$. See Section 8.1 for details.

For m-points V-statistics, $V_{m,w}^{(n)}$, the asymptotic analysis requires a more careful treatment. Deriving their weak limit involves a stronger summability condition, resembling (*c2*) used in Sobolev statistics. The proof builds on the convergence of the normalized average of vectors of spherical harmonics. To express $V_{m,w}^{(n)}$ in those terms, we apply the following interchange:

$$\begin{aligned} V_{m,w}^{(n)} &= n^{-m/2} \sum_{i_1,\ldots,i_m=1}^{n} \sum_{k=1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \psi_{k,r}\left(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}\right) \\ &= n^{-m/2} \sum_{k=1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \sum_{i_1,\ldots,i_m=1}^{n} \prod_{j=1}^{m} g_{k,r}(\mathbf{X}_{i_j}) \\ &= \sum_{k=1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \left(n^{-1/2} \sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i) \right)^m . \end{aligned} \tag{11}$$

Note that the statistic is nonnegative for even values of m.

A sufficient condition to ensure the kernel (7) converges is that w is such that

$$\sum_{k=1}^{\infty} |w_k|\, d_{q,k}^{(m-\delta_{m2}+1)/2} < \infty. \tag{12}$$

While this condition for $m = 2$ comes from the addition formula for spherical harmonics (8) and the bound $\sup_{x\in[-1,1]} \left|C_k^{(q-1)/2}(x)\right| \leq C_k^{(q-1)/2}(1)$, and coincides with the summability condition (*c2*) of Sobolev test statistics, the sufficient condition for $m > 2$ arises from

$$\left|\sum_{r=1}^{d_{q,k}} \prod_{j=1}^{m} g_{k,r}(\mathbf{X}_j)\right| \leq \left(\sum_{r=1}^{d_{q,k}} \prod_{j=1}^{m-1} g_{k,r}^2(\mathbf{X}_j)\right)^{1/2} \left(\sum_{r=1}^{d_{q,k}} g_{k,r}^2(\mathbf{X}_m)\right)^{1/2} \leq d_{q,k}^{(m+1)/2},$$

where we used (8) and the bound on $C_k^{(q-1)/2}$, yielding $|\sum_{r=1}^{d_{q,k}} g_{k,r}(\mathbf{x}) g_{k,r}(\mathbf{y})| \leq d_{q,k}$, and $\sum_{r=1}^{d_{q,k}} \prod_{j=1}^{m} g_{k,r}^{2\ell}(\mathbf{x}_j) \leq d_{q,k}^{1+m\ell}$ which holds for $m \geq 1$ and $\ell \geq 1$, and can be proven by induction using the sup-norm bound of spherical harmonics $\sup_{\mathbf{x}\in\mathbb{S}^q} |g_{k,r}(\mathbf{x})| \leq \sqrt{d_{q,k}}$. We have the following result for V-statistics, supporting the construction of tests based on $V_{m,w}^{(n)}$.

Theorem 4.2. *Let $q \geq 1$, $m \geq 2$ be an even integer, and w a real nonnegative sequence that fulfills condition* (12)*. Then, under $\mathcal{H}_0$ and as $n \to \infty$,*

$$V_{m,w}^{(n)} \rightsquigarrow V_{m,w}^{\infty} := \sum_{k=1}^{\infty} \sum_{r=1}^{d_{q,k}} w_k Z_{k,r}^m. \tag{13}$$

Remark 4.1. *Once $V_{m,w}^{(n)}$ is expressed in the form of* (11)*, a sufficient condition for the convergence* (13) *to hold is $\sum_{k=1}^{\infty} w_k\, k^{q+\lambda_q(m)-1} < \infty$ where $\lambda_q(\cdot)$ is defined in Lemma 4.1 below. This requirement is milder than the summability condition* (12)*. It arises from applying L_p asymptotic bounds for spherical harmonics to guarantee the uniform convergence of the distribution of the K-finite statistic to that of the infinite one at a specific sample size n as $K \to \infty$, for all sufficiently large n.*

Lemma 4.1 (Sogge (1986)'s L_p asymptotic bound)**.** *Let $q \geq 1$, $k \geq 1$, $1 \leq r \leq d_{q,k}$, and $\ell \geq 2$. Define*

$$\ell_q := \begin{cases} \infty, & q = 1, \\ \frac{2(q+1)}{q-1}, & q > 1 \end{cases} \quad \text{and} \quad \lambda_q(\ell) := \begin{cases} \frac{q-1}{2}\left(\frac{\ell}{2} - 1\right), & 2 \leq \ell < \ell_q, \\ q\left(\frac{\ell}{2} - 1\right) - \frac{\ell}{2}, & \ell_q \leq \ell < \infty. \end{cases}$$

Then, $\mathrm{E}\left[|g_{k,r}(\mathbf{X})|^\ell\right] = O\left(k^{\lambda_q(\ell)}\right)$.

5 Non-null asymptotics

In this section, we obtain the asymptotic behavior of m-points U- and V-statistics under three alternative scenarios: fixed alternatives, $\sqrt{n}$-local alternatives with a general form, and rotationally symmetric local alternatives with concentration $\kappa_n \to 0$. We still use the notation from Section 4 for $\{Z_{k,r}\}$, H_m, and $\rightsquigarrow$, and we use $\stackrel{\mathrm{P}}{\to}$ for convergence in probability.

5.1 Consistency against fixed alternatives

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \sim H$ be an iid sample under a fixed alternative, where H has probability density function (pdf) $h \in L^2(\mathbb{S}^q, \nu_q)$ with respect to the uniform measure ν_q that admits the expansion $h(\mathbf{x}) := 1 + h_0(\mathbf{x})$ with $h_0(\mathbf{x}) := \sum_{k=1}^{\infty} \mathbf{h}_k' \mathbf{g}_k(\mathbf{x})$, where $\mathbf{h}_k \in \mathbb{R}^{d_{q,k}}$ and $\mathbf{g}_k : \mathbb{S}^q \to \mathbb{R}^{d_{q,k}}$ is defined as $\mathbf{g}_k := (g_{k,1}, \ldots, g_{k,d_{q,k}})'$. We assume that $\sum_{k=1}^{\infty} \mathbf{h}_k' \mathbf{g}_k(\mathbf{x})$ converges uniformly on $\mathbb{S}^q$. The latter holds if h_0 is continuously differentiable for $q = 1$ (uniform convergence of Fourier series), and if it is $\lfloor q/2 \rfloor$-times continuously differentiable for $q \geq 2$, see Theorem 1 in Kalf (1995).

The following result shows the consistency of m-points U- and V-statistics for an even m and a positive weight sequence. Note that the distribution H must be such that $\mathbf{h}_k$ is non-null for at least one $k \geq 1$ to represent an alternative distribution different to the uniform.

Proposition 5.1. *Let $q \geq 1$, $m \geq 2$ be an even integer, and w a real sequence such that $\sum_{k=1}^{\infty} w_k^2 d_{q,k} < \infty$ and $w_k > 0$ for all $k \geq 1$. Let $\mathcal{S}_{\neq} = \{(k, r) : h_{k,r} \neq 0\}$, where $h_{k,r}$ denotes the rth element of $\mathbf{h}_k$. Assume $\mathcal{S}_{\neq}$ is non-empty. Then, under H and as $n \to \infty$:*

(i) $U_{m,w,K}^{(n)} \stackrel{\mathrm{P}}{\to} +\infty$ *and* $V_{m,w,K}^{(n)} \stackrel{\mathrm{P}}{\to} +\infty$, *for* $K > \min\{k : (k, r) \in \mathcal{S}_{\neq}\}$;

(ii) $V_{m,w}^{(n)} \stackrel{\mathrm{P}}{\to} +\infty$, *given that w fulfills* (12);

(iii) $U_{m,w}^{(n)} \stackrel{\mathrm{P}}{\to} +\infty$, *under the assumption of $\mathcal{S}_{\neq}$ being finite.*

When any of the assumptions on the parity of m or positiveness of w is dropped, the analysis becomes more convoluted, even for K-finite statistics. For instance, in the case of V-statistics with even m and $w_k < 0$ for some $k \geq 1$, $V_{m,w,K}$ could also diverge to $-\infty$ with non-zero probability depending on w and $\{\mathbf{h}_k\}$. The same could happen for an odd m and a positive sequence w. In the case where m is even and $w_{k_*} = 0$ for some $k_* \geq 1$, a counterexample in which $h_{k_*} \neq 0$ and $h_k = 0$ for all $k \neq k_*$ suffices to show $V_{m,w} = O_{\mathrm{P}}(1)$, and thus consistency does not hold.

5.2 Asymptotics under general local alternatives

Let $h \in L^2(\mathbb{S}^q, \nu_q)$ be a pdf with respect to the uniform measure ν_q, such that it admits the expansion $h(\mathbf{x}) := 1 + h_0(\mathbf{x})$, where h_0 is defined as in Section 5.1. The sequence of $\sqrt{n}$-local alternatives is given by the pdf h_n, with respect to the Lebesgue measure σ_q, through

$$h_n(\mathbf{x}) := \frac{1}{\omega_q} \left\{ 1 + n^{-1/2} \sum_{k=1}^{\infty} \mathbf{h}_k' \mathbf{g}_k(\mathbf{x}) \right\}, \quad \mathbf{x} \in \mathbb{S}^q. \tag{14}$$

Note that we can rewrite

$$h_n(\mathbf{x}) = \frac{1}{\omega_q} \left(1 - n^{-1/2} \right) + n^{-1/2} \frac{h(\mathbf{x})}{\omega_q}, \tag{15}$$

which exposes the structure of this sequence of alternatives. Specifically, h_n consists of a uniform component and a perturbation from uniformity vanishing at rate $n^{-1/2}$.

The next result provides the asymptotic distribution under local alternatives for finite m-points U- and V-statistics. This distribution corresponds to a shifted version of the corresponding null asymptotic distribution, with the shift terms characterizing the power of the tests introduced in Section 6. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \sim h_n$ be an iid sample, and assume that $\sum_{k=1}^{\infty} \mathbf{h}_k' \mathbf{g}_k(\mathbf{x})$ converges uniformly on $\mathbb{S}^q$.

Proposition 5.2. *Let $q \geq 1$, $K \geq 1$, $m \geq 2$, and w be a real sequence. Then, under h_n and as $n \to \infty$:*

(i) $U_{m,w,K}^{(n)} \rightsquigarrow U_{m,w,K}^{(h,\infty)} := \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k H_m \left(Z_{k,r} + h_{k,r}\right);$

(ii) $V_{m,w,K}^{(n)} \rightsquigarrow V_{m,w,K}^{(h,\infty)} := \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k \left(Z_{k,r} + h_{k,r}\right)^m;$

where $h_{k,r}$ denotes the rth element of $\mathbf{h}_k$ in both (i) and (ii).

Remark 5.1. *For $m = 2$, the asymptotic distributions under h_n can be written as*

$$U_{2,w,K}^{(h,\infty)} = \sum_{k=1}^{K} w_k \chi^2_{d_{q,k}}\left(\|\mathbf{h}_k\|^2\right) - \sum_{k=1}^{K} w_k d_{q,k} \quad \text{and} \quad V_{2,w,K}^{(h,\infty)} = \sum_{k=1}^{K} w_k \chi^2_{d_{q,k}}\left(\|\mathbf{h}_k\|^2\right).$$

As a consequence, the asymptotic distribution under local alternatives is rotation-invariant in the case $m = 2$. To see this, consider $\mathbf{X}$ with pdf $h \in L^2(\mathbb{S}^q, \nu_q)$ given as in (14). Then, $\mathbf{OX}$, with $\mathbf{O} \in \mathrm{SO}(q+1)$ has a pdf given by

$$h^{\mathbf{O}}(\mathbf{x}) = h(\mathbf{O}^{-1}\mathbf{x}) = 1 + \sum_{k=1}^{\infty} \mathbf{h}_k' \mathbf{g}_k(\mathbf{O}'\mathbf{x}) = 1 + \sum_{k=1}^{\infty} (\mathbf{C}'\mathbf{h}_k)' \mathbf{g}_k(\mathbf{x}),$$

with $\mathbf{C}$ being an orthogonal matrix of order $d_{q,k}$ (see, e.g., Section 4.2 in Efthimiou and Frye, 2014). Therefore, the rotated alternative coefficients are $\mathbf{h}_k^{\mathbf{O}} := \mathbf{C}'\mathbf{h}_k$ and

$$\|\mathbf{h}_k^{\mathbf{O}}\|^2 = (\mathbf{C}'\mathbf{h}_k)'\mathbf{C}'\mathbf{h}_k = \|\mathbf{h}_k\|^2.$$

See Section C.2 in SM for an empirical study on the rotational invariance of the asymptotic distribution under local alternatives when $m > 2$.

To perform numerical experiments (Section 8.1), it is interesting to obtain the coefficients $h_{k,r}$ for rotationally symmetric alternatives with density $h(\mathbf{x}) = f(\mathbf{x}'\boldsymbol{\mu})$, with $\boldsymbol{\mu} \in \mathbb{S}^q$ and $f : [-1, 1] \to \mathbb{R}_{\geq 0}$. An application of the Funk–Hecke Theorem yields

$$h_{k,r} = \int_{\mathbb{S}^q} h(\mathbf{x}) g_{k,r}(\mathbf{x}) \,\mathrm{d}\nu_q(\mathbf{x}) = \int_{\mathbb{S}^q} f(\mathbf{x}'\boldsymbol{\mu}) g_{k,r}(\mathbf{x}) \,\mathrm{d}\nu_q(\mathbf{x}) = \frac{b_{q,k}(f)}{a_{q,k}}\, g_{k,r}(\boldsymbol{\mu})$$

for $k \geq 1$, where $b_{q,k}(f)$ is given by (2).

5.3 Asymptotics under rotationally symmetric local alternatives

We now consider (absolutely continuous) rotationally symmetric alternatives, that is, alternatives with pdf

$$\mathbf{x} \mapsto c_{q,\kappa,f} f(\kappa \mathbf{x}'\boldsymbol{\mu}), \tag{16}$$

with respect to the Lebesgue measure σ_q, where $\boldsymbol{\mu} \in \mathbb{S}^q$, $\kappa > 0$ is a concentration parameter, $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is an *angular function* with $f(0) = 1$ (for standardization), and $c_{q,\kappa,f}^{-1} := \omega_{q-1} \int_{-1}^{1} (1 - s^2)^{q/2-1} f(\kappa s) \,\mathrm{d}s$ is a normalizing constant. Many well-known distributions on the sphere fit in the setup of (16). For instance, the von Mises–Fisher (vMF) distributions are obtained with $f(s) = \exp(s)$. Below we write $\mathrm{P}^{(n)}_{\kappa,f}$ for the joint distribution of an iid sample of observations, $\mathbf{X}_1, \ldots, \mathbf{X}_n$, with common density (16). For any positive integer i, define the coefficients $m_{k,i}$, $k = 0, 1, \ldots, i$, through

$$\sum_{k=0}^{i} m_{k,i} C_k^{(q-2)/2}(t) = t^i.$$

The asymptotic behavior of m-points statistics is expected to be different from that of Section 5.2 since the value of f and its derivatives at zero (and only at zero) should play a role here. We have the following result for finite m-points U- and V-statistics that generalize the results by García-Portugués et al (2025) for $m = 2$. Non-trivial asymptotic power is obtained against alternatives with $\kappa_n \sim n^{-1/(2k_v)}$, where k_v is the rank of the first non-null weight coefficient, when $f^{\underline{k_v}}(0) \neq 0$ ($f^{\underline{k}}(0)$ denoting the kth derivative of f at 0). In the case $f^{\underline{k_v}}(0) = 0$, the rates of the alternatives that can be detected rely on the parity of the ranks k for which w_k and $f^{\underline{k}}(0)$ are non-zero. In the following, $\mathcal{K}_{\neq} := \{k : w_k \neq 0\}$ and the relation $k \sim \ell$ on $\mathbb{N}$ indicates that k and ℓ share the same parity.

Proposition 5.3. *Consider a real sequence w with only finitely many non-zero terms. Let f be an angular function that is $\ell(\geq K_w := \max \mathcal{K}_{\neq})$ times differentiable at zero. Let $k_w := \min \mathcal{K}_{\neq}$, fix $\tau > 0$ and let k_* be the minimum value of $k \in \{k_w, \ldots, \ell\}$ such that (a) $f^{\underline{k}}(0) \neq 0$ and (b) $\mathcal{V}_k := \{\ell = k_w, \ldots, k : \ell \sim k$ and $\ell \in \mathcal{K}_{\neq}\}$ is non-empty (assuming that such a k exists). Put moreover $k_\dagger := \min \mathcal{V}_{k_*}$. Let $q \geq 1$, $K \geq 1$, and $m \geq 2$. Then, under $\mathrm{P}^{(n)}_{\kappa_n,f}$, with $\kappa_n = n^{-1/(2k_*)}\tau$ and as $n \to \infty$:*

(i) $U^{(n)}_{m,w,K} \rightsquigarrow U^{(h,\infty)}_{m,w,K} := \sum_{k=k_w}^{K} \sum_{r=1}^{d_{q,k}} w_k H_m\left(Z_{k,r} + h_{k,r}\right)$;

(ii) $V^{(n)}_{m,w,K} \rightsquigarrow V^{(h,\infty)}_{m,w,K} := \sum_{k=k_w}^{K} \sum_{r=1}^{d_{q,k}} w_k \left(Z_{k,r} + h_{k,r}\right)^m$;

with $h_{k,r} := \dfrac{m_{k,k_*} f^{\underline{k_*}}(0)\, \tau^{k_*}}{k_*!\, a_{q,k}} g_{k,r}(\boldsymbol{\mu})\, \mathbb{1}_{\{k \sim k_*, k_\dagger \leq k \leq k_*\}}$ *in both (i) and (ii)*

Remark 5.2. *Note that, analogous to Remark 5.1, the asymptotic distribution under $\mathrm{P}^{(n)}_{\kappa_n,f}$ with $\kappa_n \sim n^{-1/(2k_*)}$ is invariant under rotations in the case $m = 2$. The reason is that $\mathbf{h}_k := \left(h_{k,1}, \ldots, h_{k,d_{q,k}}\right)$ depends on $\mathbf{g}_k(\boldsymbol{\mu})$, which under an arbitrary rotation $\mathbf{O} \in \mathrm{SO}(q+1)$ satisfies $\mathbf{g}_k(\mathbf{O}\boldsymbol{\mu}) = \mathbf{C}\mathbf{g}_k(\boldsymbol{\mu})$ with $\mathbf{C}$ an orthogonal matrix.*

6 m-points tests

In this section, we define the tests based on the m-points statistics. To do so, we determine the critical regions using the behavior under $\mathcal{H}_1$, considering the alternatives introduced in Section 5. These regions depend on both the type of statistic, U or V, and the parity of m. Throughout, we assume positive weights and allow K to be infinite.

When $m = 2$, the test is one-sided for both U- and V- statistics. This follows naturally from the equivalence between 2-points and Sobolev statistics.

For $m > 2$, the behavior differs depending on the type of statistic. First, note that for clustered observations, the kernel $\psi_{k,r}$ in (7) may take zero or negative values for certain pairs (k, r). Second, under general local alternatives, the distribution of U-statistics may shift either left or right depending on the alternative (see Section C.2 in SM). Accordingly, we define the asymptotic U-test for $m > 2$ as a two-sided test. Specifically, the m-points U-test asymptotically rejects $\mathcal{H}_0$ at significance level α when

$$U^{(n)}_{m,w,K} \geq u^{\infty}_{2,w,K,1-\alpha}, \quad \text{if } m = 2,$$

and when

$$U^{(n)}_{m,w,K} \leq u^{\infty}_{m,w,K,\alpha/2} \quad \text{or} \quad U^{(n)}_{m,w,K} \geq u^{\infty}_{m,w,K,1-\alpha/2}, \qquad \text{if } m > 2,$$

with $u^{\infty}_{m,w,K,\alpha}$ being the asymptotic α-quantile of $U^{(n)}_{m,w,K}$, obtained from Proposition 4.1.

For tests based on V-statistics, the rejection region also depends on the parity of m. Consider the case where m is even. It was shown that V-statistics are nonnegative in this case. Furthermore, under fixed alternatives, the statistic diverges in probability to $+\infty$ (see Proposition 5.1). Under local alternatives, note that

$$\mathrm{E}\left[V^{(h,\infty)}_{m,w,K}\right] = \sum_{k=1}^{K}\sum_{r=1}^{d_{q,k}} w_k \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{2j} h^{m-2j}_{k,r} \frac{(2j)!}{2^j\, j!}$$

is positive regardless of the signs of $h_{k,r}$ and increases with their magnitude. These properties justify a one-sided test. In contrast, when m is odd, the same ambiguity in the shift direction observed for U-statistics suggests a two-sided test. Thus, the V-test rejects $\mathcal{H}_0$ when

$$V^{(n)}_{m,w,K} \geq v^{\infty}_{m,w,K,1-\alpha}, \qquad \text{if } m \text{ is even},$$

and when

$$V^{(n)}_{m,w,K} \leq v^{\infty}_{m,w,K,\alpha/2} \quad \text{or} \quad V^{(n)}_{m,w,K} \geq v^{\infty}_{m,w,K,1-\alpha/2}, \qquad \text{if } m \text{ is odd},$$

with $v^{\infty}_{m,w,K,\alpha}$ being the asymptotic α-quantile of $V^{(n)}_{m,w,K}$ (Proposition 4.1).

Although two-sided tests may incur a modest efficiency loss under some alternatives, simulation results (Section 8.2) show that the increased sensitivity to non-uniformity provided by higher-order kernels more than compensates for this potential limitation.

7 Rotational invariance

The equivalence with the Sobolev class for $m = 2$ was noted in Section 3. In this particular case, m-points statistics are rotation-invariant. However, this property does not hold in general for $m > 2$. In this section, we first propose a test following a p-value aggregation of randomly-rotated tests to make the m-points test quasi-rotation-invariant for all dimensions $q \geq 1$. Then, we introduce a rotation-invariant test statistic associated with the m-points kernel, whose asymptotic distribution is readily available in the circular case.

7.1 Aggregation of randomly-rotated tests

The quasi-rotation-invariant test consists of performing R m-points tests on R randomly rotated versions of the sample, and aggregating the resulting p-values. This randomization-plus-aggregation approach builds on that of Cuesta-Albertos et al. (2009) with random projections. Since the R tests are generally dependent, we require a p-value aggregation method that remains valid under dependency. Recent work has extensively investigated such methods, particularly those based on homogeneous symmetric generalized means, defined by

$$M_{d,R}(p_1, \ldots, p_R) := \left(\sum_{r=1}^{R} p_r^d / R\right)^{1/d}, \quad d \in [-\infty, +\infty],$$

see Vovk and Wang (2020) and Vovk et al. (2022). Wilson (2019, 2020) derives significance thresholds for $M_{d,R}$ via the Generalized Central Limit Theorem (GCLT), a set of results concerning the domain of attraction of stable distributions (see, e.g., Section 17.5 in Feller, 1971). Although the GCLT approach formally assumes independence, the heavy-tailed behavior it induces grants a degree of robustness under mild dependence, particularly at small significance levels and $d \le -1$. In this work, we focus on the Harmonic Mean p-value (HMP), based on $M_{-1,R}$. This approach shows strong empirical power in our setting (see Section 8.3.1).

The test rejects $\mathcal{H}_0$ if $p_{m,w,K}^{\mathrm{HMP},R} \le \alpha$ at significance level α, where

$$p_{m,w,K}^{\mathrm{HMP},R} = \int_{1/\mathring{p}}^{\infty} f_{\mathrm{Landau}}(x; \log R + 0.874, \pi/2)\,\mathrm{d}x, \qquad \text{with} \quad \mathring{p} := M_{-1,R}(p_1, \ldots, p_R),$$

$\{p_j\}_{j=1}^R$ being the p-values of the corresponding m-points tests with weights w, and $f_{\mathrm{Landau}}(x; \mu, \sigma) = (\pi\sigma)^{-1} \int_0^\infty \exp\{-t(x-\mu)/\sigma - (2/\pi)\, t \log t\} \sin(2t)\,\mathrm{d}t$ being the Landau pdf. Note that for each defined m-points test in Section 6, there is a corresponding quasi-rotation-invariant test.

7.2 Rotation-averaged invariant test statistic

We now introduce a rotation-invariant version of the m-points test statistics. Let $q \ge 1$ and $\mathbf{O} \sim \nu_{\mathrm{SO}(q+1)}$ be a random matrix that follows the Haar distribution on $\mathrm{SO}(q+1)$. We define an m-points rotation-invariant kernel based on the real sequence w such that condition (12) is met, as

$$\begin{aligned}\tilde{\Phi}_w(\mathbf{X}_1, \ldots, \mathbf{X}_m) &:= \mathrm{E}_{\mathbf{O}}\left[\Phi_w(\mathbf{O}\mathbf{X}_1, \ldots, \mathbf{O}\mathbf{X}_m)\right] \\ &= \sum_{k=1}^{\infty} w_k \mathrm{E}_{\mathbf{O}}\left[\sum_{r=1}^{d_{q,k}} \psi_{k,r}(\mathbf{O}\mathbf{X}_1, \ldots, \mathbf{O}\mathbf{X}_m)\right] =: \sum_{k=1}^{\infty} w_k \tilde{\Phi}_{\delta_k}(\mathbf{X}_1, \ldots, \mathbf{X}_m), \end{aligned} \tag{17}$$

where Φ_w is the m-points kernel (7), and the second equality is justified by the dominated convergence theorem.

Although defined for any dimension $q \ge 1$, here we restrict to the study of the circular rotation-invariant kernel, leveraging the results obtained in Section 3.2 and using the same notation in polar coordinates. The next result gives exact expressions for a rotation-invariant kernel whose weight sequence is $\{\delta_{k\ell}\}_{k=1}^\infty$ for some $\ell \ge 1$. Notably, odd values of m make the kernel identically zero and thus ineffective. However, for even values of m, the kernel possesses a rotation-invariant component.

Lemma 7.1. *Let $m \ge 2$ and $\ell \ge 1$. Let $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_m)'$ be the polar angles of each element $(\mathbf{X}_1, \ldots, \mathbf{X}_m)'$, $\boldsymbol{\alpha} := \left(\alpha, \overset{m-2}{\ldots}, \alpha\right)$ with $\alpha \in (0, 2\pi]$, and $\mathcal{S}_{m,0} := \{\mathbf{e} \in \{1\} \times \{-1,1\}^{m-1} : s_{\mathbf{e}} = 0\}$ where $s_{\mathbf{e}} := \sum_{i=1}^m e_i$ with $\mathbf{e} = (e_1, \ldots, e_m)'$. Let $\tilde{\Phi}_{m,\delta_\ell}(\boldsymbol{\theta}) := \tilde{\Phi}_{\delta_\ell}(\mathbf{X}_1, \ldots, \mathbf{X}_m)$. Then,*

$$\tilde{\Phi}_{m,\delta_\ell}(\boldsymbol{\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{m,\delta_\ell}(\boldsymbol{\theta} + \boldsymbol{\alpha})\,\mathrm{d}\alpha = \begin{cases} 2^{2-m/2} \sum_{\mathbf{e} \in \mathcal{S}_{m,0}} \cos\left(\ell \mathbf{e}'\boldsymbol{\theta}\right), & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases} \tag{18}$$

The invariant kernel $\tilde{\Phi}_{m,\delta_\ell}$ can be projected onto the basis of $L^2((\mathbb{S}^2)^m)$ as in (4), yielding the corresponding sequence of weights.

Proposition 7.1. *Let $q = 1$, $\ell \ge 1$, $m \ge 2$ an even integer, and $\mathcal{S}_{m,0}$ be as defined in Lemma 7.1. Let $\mathbf{k} = (k_1, \ldots, k_m)$, $\mathbf{r} = (r_1, \ldots, r_m)$, with $k_j \ge 1$ and $r_j \in \{1, 2\}$ for $1 \le j \le m$, and S be the number of elements in $\mathbf{r}$ equal to 1, i.e., $S = \sum_{j=1}^m \mathbb{1}_{\{r_j = 1\}}$. Let $u_{\mathbf{k},\mathbf{r}} = \langle \tilde{\Phi}_{m,\delta_\ell}, g_{k_1,r_1} \cdots g_{k_m,r_m} \rangle_m$. Then, the following statements hold:*

(i) If $\mathbf{k} \ne (\ell, \ldots, \ell)$, then $u_{\mathbf{k},\mathbf{r}} = 0$ for all $\mathbf{r}$.

(ii) If S is odd, then $u_{\mathbf{k},\mathbf{r}} = 0$ for all $\mathbf{k}$.

(iii) If $\mathbf{k} = (\ell, \ldots, \ell)$ and S is even, then:

(a) $u_{\mathbf{k},\mathbf{r}} = 2^{2-m}\binom{m-1}{m/2}$ for $S \in \{0, m\}$;

(b) $u_{\mathbf{k},\mathbf{r}} = 2^{2-m}(-1)^{S/2}\left[2\left(\sum_{0 \le t \le m/4} \binom{S}{2t}\binom{m-1-S}{m/2-2t}\right) - \binom{m-1}{m/2}\right]$, for $0 < S < m$.

Notably, imposing rotational invariance produces some nonzero weights for indices $\mathbf{r} \notin \{(1, \ldots, 1), (2, \ldots, 2)\}$, thus breaking condition $(w1)$, and resulting in a non-diagonal m-points kernel. Consequently, it is not possible to construct an m-points statistic based on $\tilde{\Phi}_w$. Nonetheless, following the general framework introduced in Section 3, we can define an associated finite V-statistic.

Definition 7.1. *Let $q = 1$, $K \geq 1$, $m \geq 2$ be an even integer, and w be a real sequence. Then, the K-finite m-points rotation-invariant V-statistic associated with w is defined as*

$$\tilde{V}_{m,w,K}^{(n)} := n^{-m/2} \sum_{i_1,\ldots,i_m=1}^{n} \tilde{\Phi}_{w,K}(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}),$$

based on the invariant kernel $\tilde{\Phi}_{w,K}(\mathbf{X}_1, \ldots, \mathbf{X}_m) = \sum_{k=1}^{K} \sum_{\mathbf{r}} v_{k,S_\mathbf{r}} \prod_{j=1}^{m} g_{k,r_j}(\mathbf{X}_j)$, with $S_\mathbf{r} = \sum_{j=1}^{m} \mathbb{1}_{\{r_j=1\}}$ and $v_{k,S} := w_k u_S$, where u_S is given by:

(i) If S is even:

(a) $u_S = 2^{2-m}\binom{m-1}{m/2}$ for $S \in \{0, m\}$;

(b) $u_S = 2^{2-m}(-1)^{S/2}\left[2\left(\sum_{0\leq t\leq m/4}\binom{S}{2t}\binom{m-1-S}{m/2-2t}\right) - \binom{m-1}{m/2}\right]$ for $0 < S < m$.

(ii) Otherwise, $u_S = 0$.

In this last result, we provide the null asymptotic distribution of the rotation-invariant V-statistic, which can be used to construct asymptotic tests of uniformity.

Proposition 7.2. *Let $q = 1$, $K \geq 1$, $m \geq 2$ be an even integer, and w be a real sequence. Then, under $\mathcal{H}_0$ and as $n \to \infty$,*

$$\tilde{V}_{m,w,K}^{(n)} \rightsquigarrow \tilde{V}_{m,w,K}^{\infty} := \sum_{k=1}^{K} \sum_{s=0}^{m/2} \binom{m}{2s} v_{k,2s} Z_{k,1}^{2s} Z_{k,2}^{m-2s},$$

with $v_{k,S}$ given in Definition 7.1.

8 Numerical experiments

This section empirically illustrates the theoretical properties of the proposed tests and highlights the advantages of m-points tests over the Sobolev class of tests. It is divided into three parts. Section 8.1 compares the V- and U-tests in terms of computational cost, behavior under the null, and performance under local alternatives. Due to the computational efficiency and retained power gains, the remaining sections focus on V-tests. Section 8.2 presents extensive simulations evaluating the power of V-tests under various fixed alternative scenarios, demonstrating improvements over Sobolev tests. Finally, Section 8.3 investigates the power and empirical size of the quasi-rotation-invariant m-points V-tests introduced in Section 7.1, along with an analysis of the asymptotic distribution of the rotation-invariant V-test statistic from Section 7.2.

In all subsequent sections, the tests defined in Section 6 are conducted at significance level $\alpha = 5\%$, unless otherwise stated, and using specific weight sequences w derived from the classical Sobolev tests—namely, Cramér–von Mises (equivalent to Watson when $q = 1$), Anderson–Darling, and Smooth Maximum. The latter coincides with the Rayleigh (1919) test as $\kappa \to 0$. The corresponding weights are listed in Table 1. Unless specified otherwise, Monte Carlo simulations use $M = 10^4$ samples.

Legend	Test	$q=1$	$q=2$
CvM	Cramér–von Mises/Watson	$(\sqrt{2}\pi k)^{-2}$	$(2(2k+3)(2k-1)(2k+1))^{-1}$
AD	Anderson–Darling	$(\sqrt{2\pi}k)^{-2}\int_0^\pi \frac{1-\cos(2k\theta)}{(\pi-\theta)\theta}\,\mathrm{d}\theta$	$(k(k+1)(2k+1))^{-1}$
S(κ)	Smooth Maximum, $\kappa>0$	$e^{-\kappa}\mathcal{I}_k(\kappa)$	$\sqrt{\pi/(2\kappa)}\,e^{-\kappa}\mathcal{I}_{k+1/2}(\kappa)$

Table 1: Weight sequences w used for m-points tests simulations when $q \in \{1,2\}$. See García-Portugués et al. (2023) and Fernández-de-Marcos and García-Portugués (2023) for weight sequences when $q>2$.

8.1 Comparison of V- and U-tests

8.1.1 Computational cost

The computational cost of U-statistics is strongly influenced by m. Specifically, the U-statistic evaluates the kernel over all possible m-ordered tuples of observations taken from a sample of size n, which results in $\binom{n}{m}$ evaluations. Meanwhile, a naïve computation of the V-statistic evaluates the kernel on every possible m-tuple with repetitions, which increases the previous count to n^m. However, the transformed version of the V-statistic (11)—essential for the derivation of the asymptotic behavior of infinite kernels—plays a key role in reducing the computational cost. Following the same steps, for the K-finite V-statistic we obtain

$$V_{m,w,K}^{(n)} = \sum_{k=1}^{K} w_k \sum_{r=1}^{d_{q,k}} \left(n^{-1/2} \sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i) \right)^m,$$

which entails a computation of $O(K\, d_{q,K}\, n)$. Surprisingly, the cost of this transformed V-statistic is independent of m, making it highly advantageous in practice. Note this transformation also reduces the cost of evaluating truncated kernels of Sobolev tests, which entail a computation of order $O(n^2)$, obviously at the expense of some (arbitrarily small) discrepancy with the non-truncated kernel. Table 2 compares the average computation time of finite U- and V-statistics across different values of m and q. The results clearly demonstrate that the computational advantage of V-statistics becomes increasingly pronounced as m grows.

m	$q=1$		$q=2$		$q=3$	
	V (ms)	U ($\times$)	V (ms)	U ($\times$)	V (ms)	U ($\times$)
2	1.7	×1	8.2	×1	25.1	×1
3	1.7	×2	7.9	×1	25.5	×1
4	1.7	×8	8.1	×3	25.0	×2
5	1.8	×36	8.3	×11	25.4	×5
6	1.8	×161	8.4	×47	26.4	×21

Table 2: Relative mean computational times for $U_{m,w,K}$ for a sample of size $n=30$, compared to the corresponding $V_{m,w,K}$ computational times (in milliseconds). All statistics use an arbitrary positive sequence w with $K=5$. Average times are estimated with 10^3 Monte Carlo samples.

8.1.2 Empirical sizes

This section compares the empirical sizes of the m-points U- and V-tests on finite samples using asymptotic critical values. We conducted a Monte Carlo simulation with $M=10^3$ samples drawn under $\mathcal{H}_0$ on $\mathbb{S}^q$, for sample sizes $n \in \{30,50,100\}$ and dimensions $q \in \{1,2,3\}$. For each sample, we performed the tests using $V_{m,w,10}$, $U_{m,w,10}$, and its corrected version $U^*_{m,w,10}$ (see Section 4.2), for $m \in \{2, 3, 4, 5, 6\}$ and weight sequences w specified in Table 1. The asymptotic critical values were approximated by simulating 10^6 Monte Carlo samples of the set of iid standard normal random variables $\{Z_{k,r}\}_{k=1,r=1}^{K_{\max},d_{q,k}}$, $K_{\max} = 10$, according to Proposition 4.1.

The empirical rejection proportions are shown in Table 3. V-tests generally maintain the significance level even for relatively small sample sizes (e.g., $n = 30$). In contrast, U-tests tend to be miscalibrated, and this effect persists even at $n = 100$, especially for larger values of m. For this reason, the corrected U^*-test is proposed. This correction substantially improves the calibration for all m. However, for higher values $m = 5, 6$, the U^*-tests appear to be significantly more conservative than V-tests. This discrepancy, which vanishes for larger sample sizes (e.g., $n = 100$), may be attributed to additional terms that vanish asymptotically but are not included in the scaling correction in U^*.

8.1.3 Power under local alternatives

In this section, we investigate the asymptotic power attained by m-points U- and V-tests under the local alternatives introduced in Section 5.2. The scenarios explored in dimensions $q \in \{1, 2, 3\}$ are in the form of (15) where the pdf h corresponds to:

- ($\vee$) *vMF* ($q = 1, 2, 3$): the vMF distribution with concentration $\kappa > 0$ and location parameter $\boldsymbol{\mu} = (0, \ldots, 0, 1)' \in \mathbb{S}^q$. Its pdf $f_{\boldsymbol{\mu},\kappa} : \mathbb{S}^q \to \mathbb{R}$ is given by $\mathbf{x} \mapsto c_{q,\kappa} e^{\kappa \mathbf{x}'\boldsymbol{\mu}}$, where $c_{q,\kappa}$ is the corresponding normalizing constant.

- ($\circ$) *Belts* ($q = 2$): a mixture of $2N$ equally-weighted vMF distributions whose locations $\{\pm\boldsymbol{\mu}_1, \ldots, \pm\boldsymbol{\mu}_N\} \subset \mathcal{S}_\theta := \{(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)' : \phi \in [0, 2\pi)\}$ for a given $\theta \in [0, \pi/2)$, are equally spaced in $\mathcal{S}_\theta$ with $\boldsymbol{\mu}_1 = (\sqrt{1 - \cos^2\theta}, 0, \cos\theta)'$, and share a common concentration κ.

- ($+$) *Cross-like* ($q = 1, 2, 3$): a mixture of $2(q+1)$ equally-weighted vMF distributions with locations $\boldsymbol{\mu}_j := (-1)^j \mathbf{e}_{\lfloor (j+1)/2 \rfloor}$ for $1 \leq j \leq 2(q + 1)$, where $\mathbf{e}_j$ denotes the jth canonical vector, and with common concentration κ.

For each local alternative scenario, we computed the asymptotic distribution given by Proposition 5.2 by simulating 10^6 Monte Carlo samples of the sets of iid standard normal random variables $\{Z_{k,r}\}_{k=1,r=1}^{K_{\max},d_{q,k}}$, $K_{\max} = 10$. The asymptotic critical values were also computed by the same method, as described in Section 8.1.2. Then, the asymptotic powers of U- and V-tests for $m \in \{2, 3, 4, 5, 6\}$ and weights w specified in Table 1 were computed and presented in Table 4. This table shows that, for $m > 3$, V-tests exhibit superior power compared to their U-test counterparts. In contrast, for $m = 3$, U-tests seem to outperform V-tests. For $m = 2$, both tests have identical power, as the asymptotic distribution of the U-statistic is simply a shifted version of the V-statistic distribution.

8.2 Power under fixed alternatives

An extensive empirical study analyzes the power attained by m-points V-tests compared to their $m = 2$ equivalent Sobolev tests under several scenarios. The simulated scenarios depend on the dimension explored, but, in contrast to the previous section, they are fixed alternatives. In the case $q = 1$, the scenario consists on

- ($*$) (*MvMF*) a mixture of N equally-weighted vMF distributions whose locations $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_N$ are equally spaced along $\mathbb{S}^1$, with $\boldsymbol{\mu}_1 = \mathbf{e}_1 = (1, 0)'$, and share a common concentration κ;

in the case $q = 2$, we explore Belts ($\circ$) and Cross-like ($+$) alternatives.

Under each specific scenario and for each sample of size $n = 100$, we performed V-tests using $V_{m,w,10}$ for $m \in \{2, 3, 4, 5, 6\}$ and w listed in Table 1. Asymptotic critical values were computed as in Section 8.1.2, and empirical rejection proportions were obtained. The goal of these experiments is to quantify the power advantage achieved by using larger m compared to the baseline Sobolev

m	Test	n	$q=1$				$q=2$				$q=3$			
			CvM	AD	S(0.1)	S(10)	CvM	AD	S(0.1)	S(10)	CvM	AD	S(0.1)	S(10)
2	V	30	5.0	4.9	5.0	4.8	5.9	6.1	5.0	5.4	5.3	5.3	5.0	5.8
		50	5.9	5.9	6.0	5.3	4.8	4.6	4.6	4.1	**3.4**	**3.3**	**3.4**	4.8
		100	3.9	4.3	**3.7**	5.2	4.0	4.5	4.0	4.9	4.4	4.4	4.2	5.2
	U	30	5.0	5.7	6.4	5.2	5.6	5.5	**7.1**	6.0	**6.9**	**6.6**	**7.3**	4.6
		50	5.6	4.5	6.3	5.8	4.2	4.0	5.2	3.9	5.4	5.5	5.4	4.4
		100	4.8	5.2	6.0	4.8	6.1	**6.5**	5.5	5.4	4.6	4.7	4.6	4.4
	U^*	30	**3.6**	**3.6**	4.2	**3.4**	4.6	4.8	5.2	4.5	5.2	5.2	5.0	4.5
		50	4.8	4.9	5.4	4.1	4.0	4.1	4.8	4.3	4.9	5.0	4.6	5.0
		100	4.2	4.6	4.2	5.0	**3.3**	**3.7**	4.2	4.4	5.0	5.3	6.1	4.8
3	V	30	4.3	4.6	3.9	**3.5**	4.5	4.3	4.5	4.5	5.3	5.5	5.4	4.7
		50	4.8	4.3	4.5	4.5	5.5	5.4	5.1	4.7	4.9	4.6	5.1	4.7
		100	5.6	5.9	5.4	4.8	5.4	5.3	5.6	5.2	3.9	4.0	4.2	5.3
	U	30	5.7	**6.5**	5.4	5.1	**6.7**	**6.7**	**6.6**	6.4	5.6	5.4	5.4	**6.8**
		50	6.4	5.5	5.6	6.1	4.4	4.4	4.9	4.9	5.5	5.3	5.5	4.5
		100	5.0	4.1	5.1	5.4	4.4	4.5	4.4	5.5	5.2	5.3	4.9	5.9
	U^*	30	5.1	4.4	4.7	4.4	4.2	3.9	4.2	4.4	5.9	5.6	5.8	4.4
		50	4.2	4.1	5.0	4.4	5.4	5.5	5.3	5.2	**3.4**	**3.4**	**3.4**	4.0
		100	5.5	5.5	5.1	5.1	4.8	5.0	5.3	4.4	4.8	4.8	4.5	4.1
4	V	30	5.3	5.1	5.1	4.8	4.7	4.7	4.4	4.7	4.7	4.6	4.5	4.1
		50	**3.8**	3.9	4.0	4.7	4.5	4.5	4.4	4.3	5.3	5.3	5.1	5.4
		100	**3.4**	**3.6**	**3.8**	**3.7**	4.9	5.1	4.9	5.6	4.4	4.5	4.4	5.8
	U	30	**6.3**	**8.6**	**11.3**	**7.2**	**9.9**	**10.1**	**10.0**	**9.1**	**7.1**	**7.1**	**7.4**	**7.7**
		50	6.4	**8.4**	**8.7**	6.0	6.3	6.2	6.2	**6.9**	**6.5**	6.0	6.3	6.4
		100	**7.1**	5.6	**8.5**	5.6	**6.8**	6.4	6.3	**7.2**	5.5	5.5	5.5	4.9
	U^*	30	4.2	4.3	3.9	4.3	**3.6**	**3.8**	**3.3**	4.2	4.9	4.6	4.6	4.0
		50	**3.3**	**3.3**	**2.9**	**3.2**	5.0	5.1	5.7	4.6	3.9	4.2	**3.6**	3.9
		100	5.6	5.4	5.9	4.6	4.9	4.8	4.6	4.1	4.6	4.5	4.7	4.7
5	V	30	3.9	4.0	4.2	**3.8**	5.6	5.4	5.6	5.3	5.5	5.3	5.4	4.6
		50	5.8	4.0	5.8	5.8	4.8	5.1	4.5	4.2	5.5	5.3	5.5	4.2
		100	5.3	4.2	5.4	5.0	4.9	5.0	5.1	4.4	3.9	3.9	4.2	5.2
	U	30	**8.7**	**10.1**	**9.2**	**9.4**	**9.3**	**9.5**	**8.2**	**10.6**	**10.7**	**10.9**	**11.1**	**10.1**
		50	**7.1**	**8.3**	6.4	**6.6**	**7.8**	**7.6**	**8.1**	**7.8**	**10.3**	**9.8**	**10.0**	**7.7**
		100	6.0	**6.6**	6.3	**6.5**	**7.1**	**6.7**	**7.2**	**8.0**	6.3	6.3	6.1	**6.9**
	U^*	30	**1.7**	**2.2**	**2.0**	**2.6**	4.8	5.0	4.6	**3.6**	**3.2**	**3.3**	**3.4**	**3.2**
		50	4.0	4.1	4.1	**3.8**	4.3	4.1	4.3	**3.8**	5.1	4.6	4.7	4.3
		100	5.1	4.8	4.6	4.3	5.2	4.9	5.2	5.1	5.3	4.8	5.3	4.9
6	V	30	**3.0**	**3.6**	4.4	**3.5**	4.7	4.7	5.1	**3.8**	4.4	4.8	4.5	**6.6**
		50	6.3	**6.6**	5.8	5.8	5.0	4.6	4.4	5.1	4.3	4.2	4.4	5.8
		100	5.0	5.1	5.0	5.2	4.3	4.4	5.0	**3.7**	4.7	4.5	4.6	4.6
	U	30	**12.9**	**14.1**	**13.5**	**14.2**	**14.2**	**14.6**	**15.3**	**13.8**	**15.4**	**14.8**	**15.6**	**14.9**
		50	**10.7**	**7.8**	**10.4**	**12.3**	**8.6**	**8.6**	**9.0**	**11.6**	**8.7**	**9.2**	**8.3**	**10.9**
	U^*	30	**1.5**	**1.6**	**1.2**	**2.0**	**1.6**	**2.0**	**1.6**	**3.1**	**1.8**	**1.9**	**1.7**	**3.3**
		50	**2.2**	**2.4**	**1.5**	**2.6**	**2.7**	**2.9**	**2.6**	**3.1**	**2.2**	**2.3**	**2.2**	**3.6**

Table 3: Empirical sizes (%) of m-points tests ($V_{m,w,10}$, $U_{m,w,10}$, and $U^*_{m,w,10}$), with $m \in \{2,3,4,5,6\}$ and weights specified in the header row. Samples of size $n \in \{30, 50, 100\}$ are drawn in dimensions $q \in \{1,2,3\}$. Boldface indicates empirical sizes outside the 95% asymptotic confidence interval.

tests ($m = 2$). For this purpose, Figures 1–3 plot the difference in power between each m-points test and the Sobolev test as a function of the common concentration parameter κ, for each respective scenario. A summary of these results is provided in Table 6 to facilitate the discussion.

The results remain consistent across scenarios ($*$), ($\circ$), and ($+$). In general, for a fixed choice of weights w, increasing m improves the power of the test. However, such improvement occurs only when the baseline Sobolev test ($m = 2$) already achieves non-negligible power (absolute power curves

q	h_n	Test	$m=2$		$m=3$		$m=4$		$m=5$		$m=6$		κ
			V	U	V	U	V	U	V	U	V	U	
1	vMF ($\vee$)	CvM	25.3	25.3	19.5	**23.1**	**25.4**	19.6	**21.8**	17.6	**25.9**	17.5	15
		AD	26.8	26.6	18.8	**22.9**	**27.1**	19.8	**22.0**	17.2	**26.4**	17.7	15
		S(0.1)	21.4	21.4	20.5	**21.2**	**21.1**	18.0	**21.1**	16.5	**22.0**	16.3	15
		S(10)	32.4	32.2	18.1	**22.3**	**29.3**	20.4	**21.7**	17.3	**26.6**	17.4	15
	Cross (+)	CvM	5.3	**5.5**	5.1	**5.4**	5.3	**5.9**	5.6	**5.8**	**6.4**	5.6	255
		AD	5.6	**6.0**	5.3	**5.6**	5.8	**6.1**	5.9	**6.1**	**7.2**	5.9	255
		S(0.1)	5.0	5.0	4.9	5.0	4.8	**5.0**	4.8	**5.0**	4.8	4.8	63
		S(10)	8.6	8.6	8.3	**9.4**	**9.9**	8.9	**9.8**	8.9	**10.8**	7.5	1023
2	vMF ($\vee$)	CvM	28.7	28.8	21.3	**24.6**	**29.1**	22.0	**24.7**	20.4	**30.6**	18.7	15
		AD	31.0	31.0	20.6	**24.8**	**31.8**	22.8	**24.8**	20.5	**33.1**	20.1	15
		S(0.1)	24.7	24.7	23.9	**24.6**	**25.5**	20.6	**24.2**	18.2	**25.4**	17.5	15
		S(10)	46.5	46.5	20.7	**25.1**	**42.2**	23.6	**26.0**	22.3	**35.9**	22.3	15
	Belts ($\circ$) $N=6$ $\theta=\pi/4$	CvM	5.2	5.2	5.1	5.0	5.2	5.2	5.2	5.2	5.3	5.3	1023
		AD	5.3	5.2	5.2	5.2	**5.5**	5.2	5.3	5.4	**5.5**	5.0	1023
		S(0.1)	5.0	5.0	5.0	4.9	5.0	5.1	4.8	4.9	4.9	5.0	0
		S(10)	7.6	7.5	**6.7**	6.2	**7.3**	6.3	6.7	6.6	**7.0**	6.4	1023
	Cross (+)	CvM	5.5	5.4	4.9	**5.2**	**5.5**	5.0	5.1	**5.3**	5.8	5.9	1023
		AD	**5.6**	5.4	5.0	5.1	**5.6**	5.2	5.4	**6.0**	**6.4**	6.1	255
		S(0.1)	**5.1**	4.8	5.0	5.1	5.1	5.2	4.9	4.8	5.0	5.1	0
		S(10)	12.1	12.2	10.7	10.8	**14.0**	10.2	**13.1**	12.4	**15.7**	10.9	1023
3	vMF ($\vee$)	CvM	31.4	31.1	23.5	**25.9**	**32.7**	23.9	**26.6**	22.6	**34.8**	20.9	15
		AD	33.3	**33.9**	22.9	**26.4**	**35.0**	24.6	**26.6**	23.8	**36.4**	21.7	15
		S(0.1)	26.9	26.9	25.7	**26.6**	**28.2**	22.2	**26.2**	19.7	**28.2**	18.5	15
		S(10)	57.3	57.4	21.3	**26.4**	**54.4**	27.5	**29.5**	26.7	**46.0**	26.2	15
	Cross (+)	CvM	5.3	5.3	4.9	5.0	**5.3**	4.9	5.0	**5.3**	5.5	**5.7**	255
		AD	5.5	5.5	5.0	4.9	**5.5**	4.6	5.1	**5.9**	5.9	**6.5**	1023
		S(0.1)	**5.1**	4.9	5.0	4.9	5.0	4.9	5.0	5.0	**5.2**	4.8	255
		S(10)	17.1	17.1	12.8	**13.4**	**21.9**	12.9	**18.0**	16.9	**24.2**	16.5	1023

Table 4: Asymptotic power (%) of $U_{m,w,10}$- and $V_{m,w,10}$-tests, with $m \in \{2,3,4,5,6\}$ and weights specified in Table 1, under local alternatives h_n on different dimensions $q \in \{1,2,3\}$. For each alternative scenario and weight sequence, κ is chosen as the one at which the Sobolev test ($m=2$) attains the maximum power. Boldface indicates the maximum power for each combination of m and weight sequence, if the difference is significant (two-sample z-test $\alpha = 5\%$).

can be found in Figures 5–7 of Section C.1 in SM). Otherwise, increasing m does not enhance the power of the test for that particular weight sequence. Importantly, in no case explored, increasing m significantly reduces power; the power curves remain at least as high as those of the Sobolev test. These findings support viewing m-points tests as a natural extension of Sobolev tests that generally offer improved performance.

The effect of K has also been investigated. Simulations of tests based on $V_{m,w,K}$ for $K \in \{4, 10, 40\}$ with the usual values of m and w, show that increasing K does not lead to substantial changes in empirical power. However, the effect of K might depend on the decay rate of the sequence w, with higher differences for slowly decreasing weights.

8.3 Rotational invariance

8.3.1 Aggregation of randomly-rotated tests

This section demonstrates the empirical behavior of the quasi-rotation-invariant m-points tests via HMP aggregation introduced in Section 7.1. The simulations explore the empirical sizes under the null hypothesis and the power attained under alternative scenarios $(*)$, $(\circ)$, and $(+)$, already introduced in Section 8.2.

Table 5 presents the empirical sizes. For each sample of size $n = 100$ drawn under $\mathcal{H}_0$, we performed V-tests at significance levels $\alpha \in \{1\%, 5\%, 10\%\}$ based on $p^{\mathrm{HMP},R}_{m,w,10}$ with $R = 50$ random rotations, $m \in \{3, 4, 5, 6\}$, and w specified in Table 1. As discussed in Section 7.1, for lower significance levels ($\alpha < 5\%$), the tests are well calibrated. At $\alpha = 5\%$, although the empirical sizes lie outside the 95% asymptotic confidence interval, the rejection proportions do not exceed the significance level, meaning the test tends to be conservative but not liberal.

Following the simulation setting of Section 8.2, the empirical power gains of quasi-rotation-invariant V-tests, $p^{\mathrm{HMP},R}_{m,w,10}$ with $R = 50$, relative to Sobolev tests ($m = 2$) were obtained under alternative scenarios $(*)$, $(\circ)$, and $(+)$. Table 6 summarizes the comparison between m-points and the corresponding quasi-rotation-invariant tests. Observe that the latter exhibit lower power. However, the decrease is minor, and the power gains are retained. As with m-points tests, increasing m does not degrade the performance of the tests, achieving at least similar power to $m = 2$ tests. In fact, as m increases, the power gap between the m-points and the quasi-rotation-invariant tests narrows, and given the independence of computational cost from m, large-m quasi-rotation-invariant

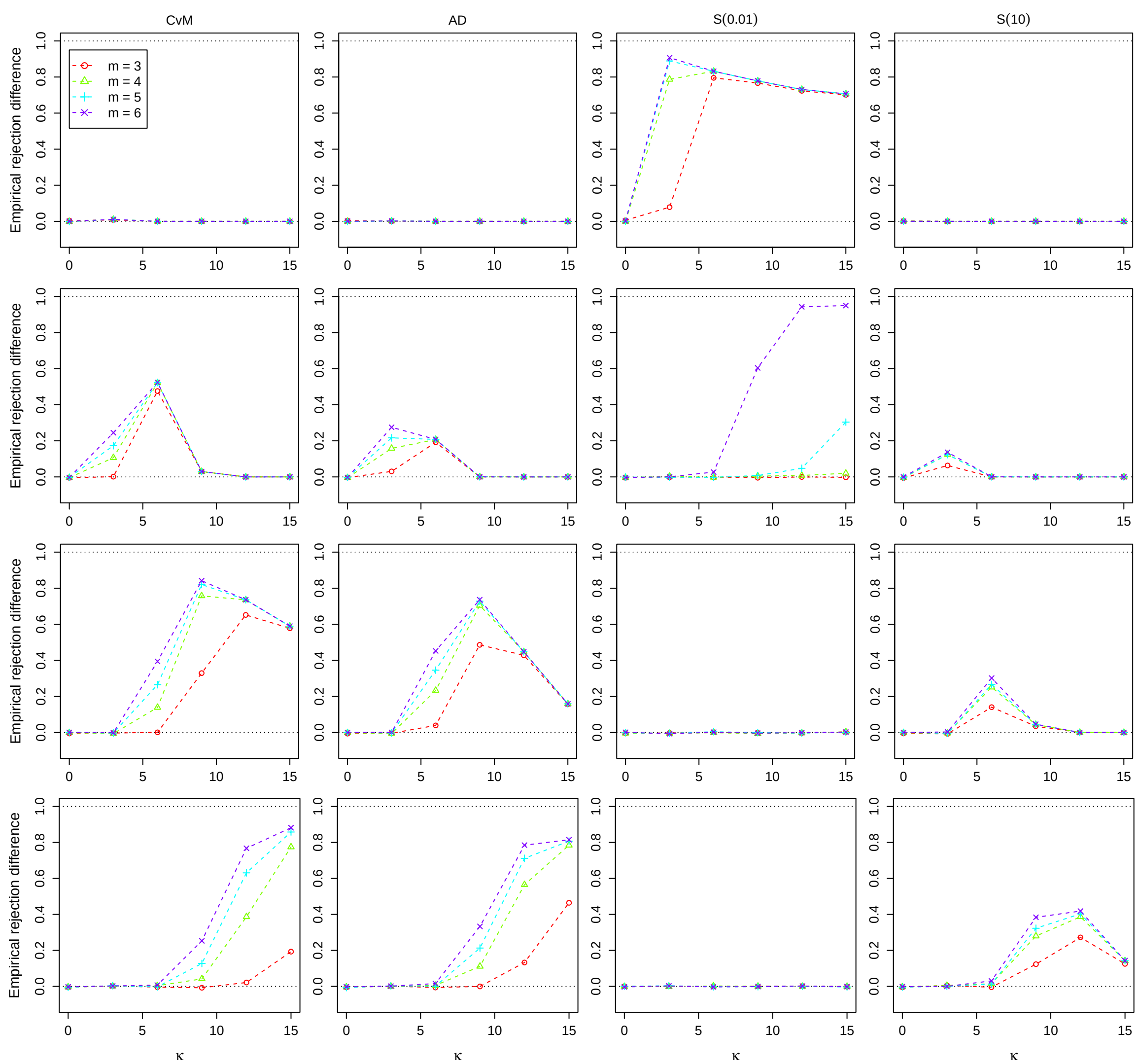

Figure 1: Empirical rejection proportion difference curves under scenario $(*)$ as a function of concentration κ. Each row corresponds to a different number of mixture components, $N \in \{2, 3, 4, 5\}$. Curves compare m-points V-tests against the baseline Sobolev test ($m = 2$). Tests are based on $V_{m,w,10}$, $m \in \{3, 4, 5, 6\}$, with weights indicated by columns, and $n = 100$.

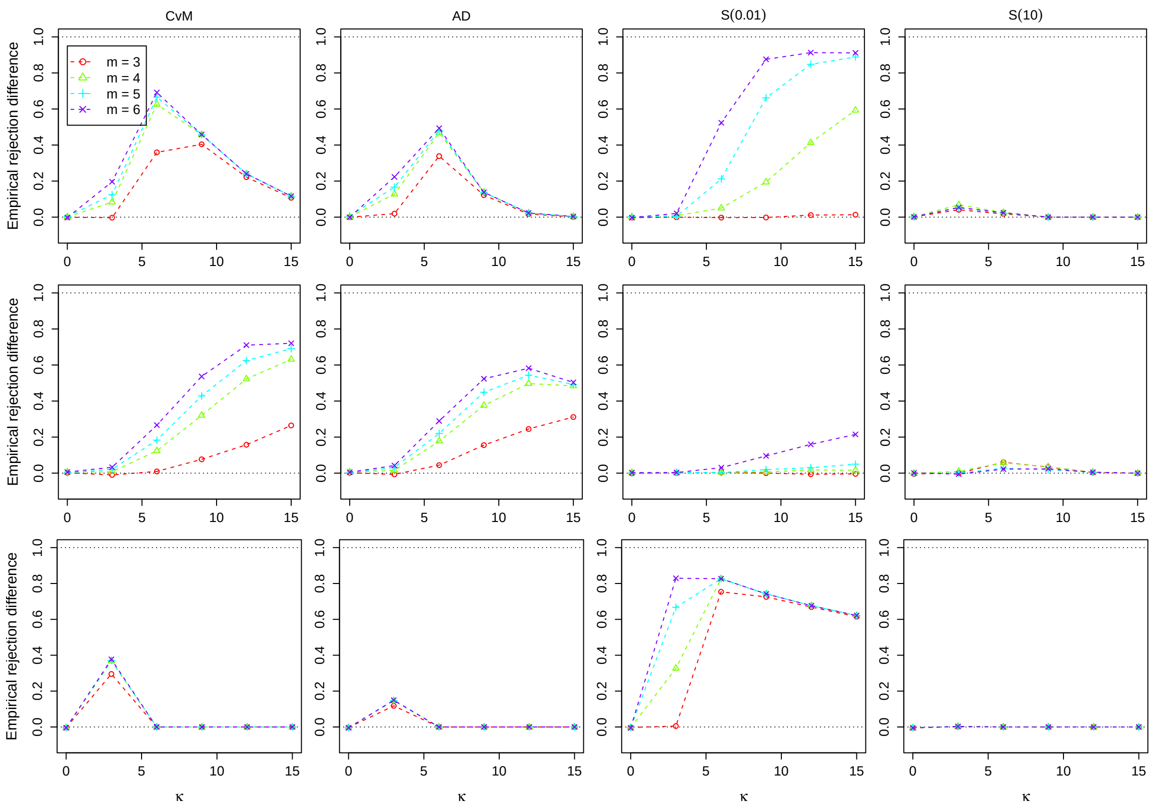

Figure 2: Empirical rejection proportion difference curves under scenario (○) with $N = 6$ as a function of concentration κ, each row corresponding to a different parameter value $\theta \in \{\pi/12, \pi/4, 5\pi/12\}$. The same description of Figure 1 applies.

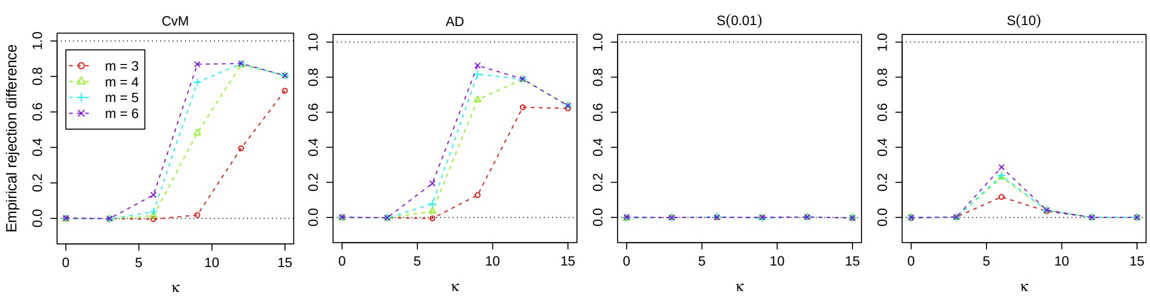

Figure 3: Empirical rejection proportion difference curves under scenario (+) as a function of concentration κ. The same description of Figure 1 applies.

V-tests remain practical. Detailed results under each alternative scenario are reported in Section C.1 in SM (Figures 8–13).

8.3.2 Rotation-averaged invariant test statistic

In this section, we illustrate the behavior of the rotation-invariant V-statistic proposed in Section 7.2. While the focus is primarily on the circular case, we also investigate the hyperspherical case using an approximation to the invariant kernel. The example consists on the m-CvM test statistic and an m-points test statistic based on an equally-weighted sequence, $w = \mathbf{1}$.

For $q = 1$, the asymptotic distribution of $V_{m,w,K}^{(n)}$ is given by $\sum_{k=1}^{K} w_k(Z_{k,1}^m + Z_{k,2}^m)$, whereas the asymptotic distribution of $\tilde{V}_{m,w,K}^{(n)}$ includes additional terms, involving the invariant coefficients $u_{k,S}$ (see Definition 7.1). Figure 4 presents P–P plots comparing the asymptotic distributions of the m-points statistic (x-axis) and its rotation-invariant counterpart (y-axis). These plots are com-

Test	q	$m=3$			$m=4$			$m=5$			$m=6$		
		1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
CvM	1	0.87	**3.77**	**6.53**	0.90	**3.57**	**7.06**	1.14	**3.99**	**6.33**	0.86	**4.23**	**6.32**
AD		1.04	**3.85**	**6.53**	1.21	**3.56**	**7.18**	**0.82**	**3.70**	**6.39**	0.93	**3.90**	**6.37**
S(0.1)		0.84	**3.75**	**6.56**	0.89	**3.62**	**7.12**	1.12	**4.12**	**6.69**	0.89	**4.24**	**6.59**
S(10)		0.84	**3.98**	**6.82**	0.89	**3.70**	**7.09**	0.95	**3.52**	**6.37**	0.96	**4.10**	**6.60**
CvM	2	1.01	**4.18**	**6.76**	0.96	**3.91**	**6.61**	1.05	**4.13**	**6.89**	0.91	**3.90**	**6.78**
AD		1.00	**4.15**	**6.74**	0.95	**3.88**	**6.51**	1.04	**4.01**	**6.87**	0.88	**3.88**	**6.81**
S(0.1)		1.03	**4.24**	**6.82**	0.95	**4.07**	**6.66**	1.04	**4.21**	**7.07**	0.89	**4.01**	**6.83**
S(10)		1.04	**4.22**	**7.51**	1.07	**4.42**	**7.05**	0.83	**4.16**	**7.17**	0.94	**4.35**	**7.63**
CvM	3	1.19	**4.13**	**6.65**	1.00	**3.88**	**6.74**	1.09	**4.06**	**7.29**	1.02	**4.18**	**7.12**
AD		1.18	**4.13**	**6.63**	0.99	**3.87**	**6.74**	1.08	**4.04**	**7.10**	1.04	**4.14**	**7.07**
S(0.1)		1.17	**4.17**	**6.67**	1.00	**3.93**	**6.71**	1.10	**4.08**	**7.36**	1.04	**4.21**	**7.12**
S(10)		**1.29**	5.04	**8.41**	1.11	4.69	**7.79**	0.99	**4.49**	**8.44**	1.16	5.38	9.45

Table 5: Empirical sizes (%) of quasi-rotation-invariant m-points V-tests ($p^{\mathrm{HMP},50}_{m,w,10}$), with $m \in \{3,4,5,6\}$ and weights in first column. Tests are conducted at $\alpha \in \{1\%, 5\%, 10\%\}$ with samples of size $n = 100$ in dimensions $q \in \{1,2,3\}$. Boldface indicates sizes outside the 95% asymptotic confidence interval.

plemented by analogous comparisons based on empirical distributions obtained from Monte Carlo samples of size $n \in \{10, 100\}$. The emphasis is on the asymptotic agreement between both versions of the statistic, while the empirical curves illustrate how closely this agreement holds in practice for moderate sample sizes. Interestingly, the differences between the m-points test statistic and its rotation-invariant version are small; indeed, they are statistically negligible ($\alpha = 5\%$) for $m = 4$, although significant for $m = 6$. These differences are mostly present in lower quantiles of the distributions, but they vanish in the upper tail. This observation is further supported in Table 7, where the empirical sizes of one-sided tests based on $\tilde{V}^{(n)}_{m,w,5}$ for $m \in \{4,6\}$ using the asymptotic critical values derived from the non-rotation-invariant statistic, suggest that the discrepancy between the two asymptotic laws is negligible in practice, in particular, for rapidly decaying weights.

For higher dimensions, $q > 1$, we approximate (17) by averaging the original non-invariant V-statistic over $R = 100$ random rotations of the sample. The two bottom rows of Figure 4 show the resulting P-P plot in $q \in \{2, 5\}$. Notably, the differences are minimal in the upper quantiles for CvM weights, but they become significant with $w = \mathbf{1}$. This may be attributed to the contribution of higher-degree spherical harmonics, whose number increases with the degree k, and when assigned a constant weight, the cumulative effect becomes more pronounced compared to rapidly decaying weight sequences. Additionally, the number of spherical harmonics involved in a given kernel increases with q, which may account for the growing differences observed as the dimension increases. Thus, the practical validity of using non-rotation-invariant asymptotic critical values for the invariant approximated V-statistic deteriorates in higher dimensions and with slowly decaying weights (Table 7).

9 Discussion

This paper provides a general class of uniformity tests on the hypersphere that extends the classical Sobolev framework while offering improved practical performance, particularly under multimodal alternatives that are challenging for traditional Sobolev tests to detect. The proposed m-points test statistics are constructed using a restricted class of square-integrable kernels of degree m, enabling a natural extension of Sobolev tests by using their corresponding sequences of weights. The strength and flexibility of this class arise from several features of the test statistics.

A key feature of the m-points statistics is their definition in two forms: based on U- and V-

q	Alternative		Test	$m=3$ V	$m=3$ HMP	$m=4$ V	$m=4$ HMP	$m=5$ V	$m=5$ HMP	$m=6$ V	$m=6$ HMP	κ
1	MvMF (∗)	$N=2$	CvM	0.8	1.0	1.1	1.1	1.1	1.1	1.1	1.1	3
			AD	0.1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	3
			S(0.1)	7.8	−2.0	78.7	59.0	88.9	84.9	90.7	89.7	3
			S(10)	0.2	−1.0	0.1	−1.0	0.1	−1.3	0.0	−1.0	0
		$N=3$	CvM	47.6	45.4	52.2	51.6	52.3	52.2	52.4	52.3	6
			AD	3.0	−1.9	15.8	8.0	21.6	13.5	27.5	19.6	3
			S(0.1)	−0.2	−1.2	2.0	0.1	30.4	−0.6	95.0	94.1	15
			S(10)	6.3	4.9	12.4	5.4	12.4	6.9	13.7	7.8	3
		$N=4$	CvM	32.9	12.1	75.8	64.8	82.0	78.8	84.1	82.2	9
			AD	48.6	35.5	70.4	65.3	72.8	71.4	73.6	72.7	9
			S(0.1)	0.3	−1.2	0.3	−1.4	0.2	−1.1	0.2	−1.3	15
			S(10)	14.0	11.5	25.1	18.0	26.7	20.9	30.2	23.9	6
		$N=5$	CvM	19.3	1.0	77.4	62.2	85.8	81.9	88.2	86.2	15
			AD	46.4	29.2	78.4	72.5	80.7	79.3	81.5	80.6	15
			S(0.1)	0.1	−0.9	−0.2	−1.2	0.3	−1.2	0.2	−1.6	3
			S(10)	27.2	25.3	38.7	34.6	40.3	37.8	41.8	39.8	12
2	Belts (◦)	$\theta=\pi/12$	CvM	35.9	11.1	62.5	53.1	66.6	62.2	69.2	67.1	6
			AD	33.9	20.6	46.8	42.6	48.0	46.2	49.3	48.3	6
			S(0.1)	1.1	−2.8	41.3	8.5	84.8	60.5	91.3	90.1	12
			S(10)	4.1	5.1	6.8	3.2	5.2	5.4	5.4	5.0	3
		$\theta=\pi/4$	CvM	26.5	−9.3	63.1	48.4	69.1	48.4	72.0	71.0	15
			AD	24.5	−7.9	49.7	38.2	54.4	36.5	58.2	55.4	12
			S(0.1)	−0.4	−1.5	1.6	−0.3	5.0	−1.3	21.5	9.2	15
			S(10)	3.3	0.2	3.5	2.9	1.9	−0.3	2.4	2.9	9
		$\theta=5\pi/12$	CvM	29.5	22.6	36.8	35.1	37.1	36.7	37.7	37.4	3
			AD	11.8	9.8	14.7	14.0	14.7	14.5	14.9	14.8	3
			S(0.1)	0.5	−3.0	32.7	12.6	66.7	43.2	82.9	73.7	3
			S(10)	0.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	3
	Cross (+)		CvM	39.5	−6.0	86.6	75.2	87.2	86.1	87.3	87.3	12
			AD	12.7	−6.3	67.1	30.5	81.7	58.2	86.7	79.3	9
			S(0.1)	−0.1	−1.2	−0.4	−0.8	−0.2	−0.4	0.2	−1.0	0
			S(10)	11.7	1.7	23.0	13.2	23.9	17.2	28.6	20.7	6

Table 6: Empirical rejection proportion difference (%) between the m-points test indicated in columns (V refering to $V_{m,w,10}$ and HMP to $p^{\mathrm{HMP},50}_{m,w,K}$ in its V-form) and the corresponding Sobolev tests ($m=2$), with weights indicated by rows, and $n=100$. For each alternative and weight sequence, κ is chosen as the one that maximizes $V_{6,w,K}$ power difference among $\kappa \in \{3k : 0 \leq k \leq 5\}$.

statistics. The structure of their critical regions depends on the form of the statistic and the parity of m. Empirical results have shown that V-tests generally outperform U-tests in terms of power, with this greater advantage when m is even. At the same time, the superior performance of V-tests is largely attributed to the truncation of kernels. Finite V-statistics (expressed in terms of spherical harmonics, unlike typical Sobolev statistics) reduce the computational complexity from $O(n^m)$ to $O(n)$, offering significant practical advantages over their U-counterparts. Nonetheless, infinite kernels can be used through the closed-form expressions obtained in the circular case, but they entail higher computational cost. The asymptotic null distribution has been derived for finite and infinite m-points statistics, and shown to be usable in practice for both U- and V-tests. Notably, infinite V-tests with even m and positive weights are omnibus under a general class of fixed alternative distributions.

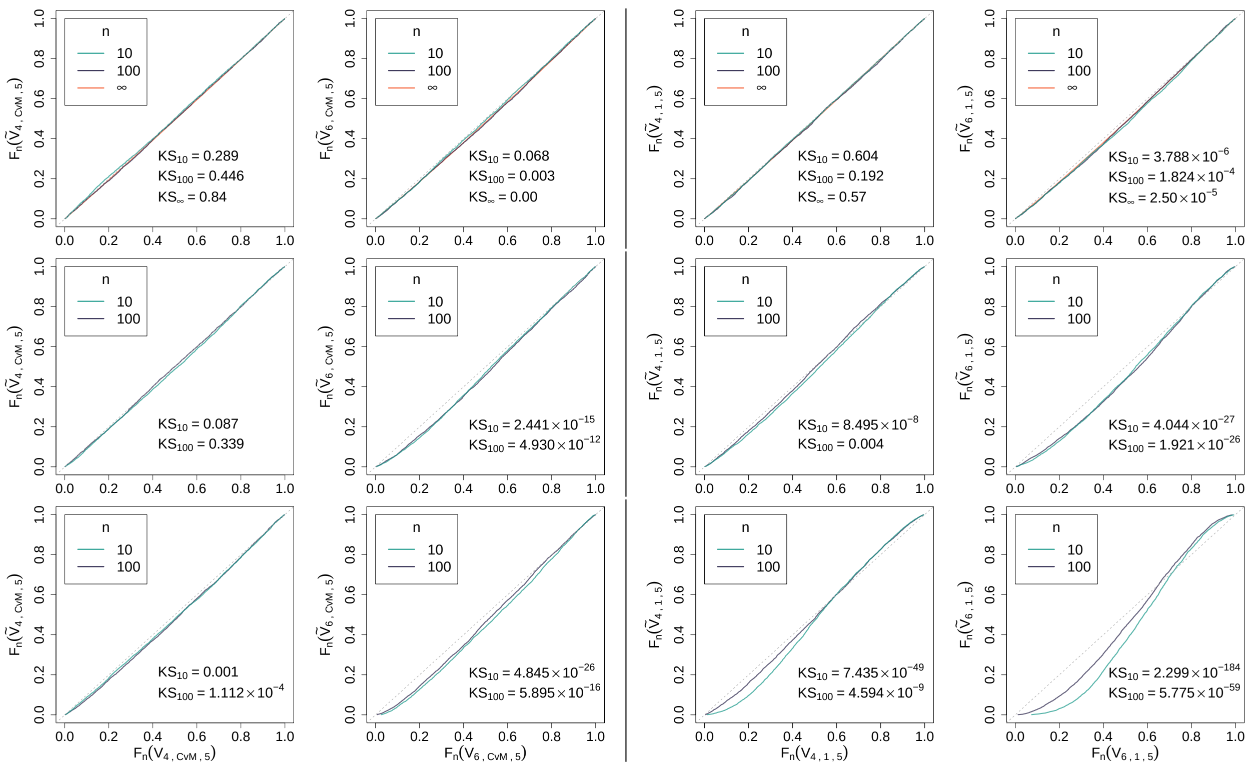

Figure 4: P-P plots comparing the asymptotic and empirical $n \in \{10, 100\}$ distributions of the m-points V-statistics, $V_{m,w,5}$, and their rotation-invariant counterparts $\tilde{V}_{m,w,5}$, for dimensions $q \in \{1, 2, 5\}$ corresponding to each row. Two choices of weights are considered: CvM (left panel) and constant $\mathbf{1}$ (right panel). Columns within each panel correspond to different values of $m \in \{4, 6\}$. p-values from two-sample Kolmogorov–Smirnov tests (KS_n) assess differences between $V_{m,w,5}^{(n)}$ and $\tilde{V}_{m,w,5}^{(n)}$.

Regarding rotational invariance, the harmonic mean p-value approach provides a quasi-invariant approach that preserves the improved performance of m-points tests over the Sobolev class. In addition, a fully rotation-invariant kernel has been introduced, with a closed-form expression for the circular case. While this kernel does not fall into the proposed m-points class, it motivates a more general form of m-points statistic, whose asymptotic results have also been derived. Interestingly, the generalized statistic is only defined for even values of m, as they exhibit some rotational invariance character, which is absent for odd m.

Supplementary materials

The supplementary materials contain the proofs of all the results presented in this paper, along with additional simulation results.

Acknowledgments

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q		Asymptotic critical values	n	$m=4$ 1%	2.5%	5%	$m=6$ 1%	2.5%	5%
1	$\tilde{V}^{(n)}_{m,\text{CvM},5}$	V	10	**0.63**	**1.93**	**4.34**	**0.56**	**1.86**	**4.15**
			100	0.88	2.39	4.90	0.87	2.36	4.90
		$\tilde{V}$	10	**0.67**	**1.97**	**4.39**	**0.62**	**1.92**	**4.12**
			100	0.93	2.44	4.95	0.94	2.42	4.88
	$\tilde{V}^{(n)}_{m,\mathbf{1},5}$	V	10	**0.38**	**1.29**	**3.03**	**0.20**	**0.94**	**2.62**
			100	0.83	**2.17**	**4.50**	**0.74**	**2.05**	**4.52**
		$\tilde{V}$	10	**0.46**	**1.42**	**3.22**	**0.28**	**1.08**	**2.82**
			100	0.96	2.34	4.80	0.90	2.28	4.77
2	$\tilde{V}^{(n)}_{m,\text{CvM},5}$	V	10	**0.58**	**1.94**	**4.38**	**0.60**	**2.01**	**4.37**
			100	0.87	2.43	4.91	0.90	2.43	4.91
	$\tilde{V}^{(n)}_{m,\mathbf{1},5}$	V	10	0.89	**1.98**	**3.82**	**0.32**	**0.98**	**2.31**
			100	**0.52**	**1.55**	**3.64**	**0.16**	**0.75**	**2.31**
5	$\tilde{V}^{(n)}_{m,\text{CvM},5}$	V	10	**0.51**	**1.87**	**4.33**	**0.40**	**1.70**	**4.44**
			100	**0.79**	**2.10**	4.60	**0.61**	**2.03**	4.79
	$\tilde{V}^{(n)}_{m,\mathbf{1},5}$	V	10	**7.39**	**12.30**	**17.84**	**8.64**	**16.90**	**26.47**
			100	**0.71**	**1.92**	**4.21**	**0.17**	**0.83**	**2.42**

Table 7: Empirical sizes (%) of one-sided m-points rotation-invariant V-tests, $\tilde{V}_{m,\text{CvM},5}$ with CvM weights (Table 1), and $\tilde{V}_{m,\mathbf{1},5}$ with $w=\mathbf{1}$, for $q \in \{1,2,5\}$. Tests are conducted at $\alpha \in \{1\%, 2.5\%, 5\%\}$ using two different asymptotic critical values: those of the corresponding non-rotation-invariant statistics (V), $V_{m,\text{CvM},5}$ and $V_{m,\mathbf{1},5}$, and those of the rotation-invariant statistics ($\tilde{V}$). Boldface indicates empirical sizes outside the 95% asymptotic confidence interval.

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Supplementary materials for "On m-points uniformity tests on hyperspheres"

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Abstract

These supplementary materials are divided into three parts. Section A contains the proofs of the results presented in the paper, Section B gathers and proves technical lemmas, and Section C includes additional results obtained from numerical experiments.

Keywords: Circular data; Spherical data; Uniformity tests.

A Proofs of the main results

A.1 Proofs of Section 3

Proof of Proposition 3.1. Note that using the notation in Lemma B.1, for a general sequence of weights $w = \{w_k\}_{k=1}^{\infty}$, we can write its associated kernel as the series

$$\Phi_w(\mathbf{X}_1, \dots, \mathbf{X}_m) = \sum_{k=1}^{\infty} w_k \Phi_{\delta_k}(\mathbf{X}_1, \dots, \mathbf{X}_m),$$

with $\delta_k = \{\delta_{jk}\}_{j=1}^{\infty}$, which by Lemma B.1 for even values of m results in the desired expression. □

Proof of Corollary 3.1. The proof relies on straightforward computation and extension by symmetry explained in Section 3.2, once the trigonometric series arising in Fourier series, are identified. The expression for $\Phi_{m,w_{\mathrm{W}}}$ is directly obtained through the well known identity

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k^2} = \pi^2 B_2(\{x\}), \quad x \in \mathbb{R},$$

and $\Phi_{m,w_{\mathrm{P}}}$ using the identity

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} = -\log\left(2\sin\frac{\theta}{2}\right), \quad 0 < \theta < 2\pi.$$

The rest are obtained from the following works: $\Phi_{m,w_{\mathrm{R}}}$ from Corollary 3.6 and Proposition 2.6 in García-Portugués et al. (2023), $\Phi_{m,w_{\mathrm{AD}}}$ from Propositions 2.7 and 3.2 in the same paper, and $\Phi_{m,w_{\mathrm{S}}}$ and $\Phi_{m,w_{\mathrm{Pois}}}$ from Proposition 3 in Fernández-de-Marcos and García-Portugués (2023). □

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A.2 Proofs of Section 4

Proof of Proposition 4.1. We first prove part (i). Let $k \geq 1$ and $1 \leq r \leq d_{q,k}$ be fixed. We make use of the hollow sum formula given in Rubin and Vitale (1980), which holds for the kernel $\psi_{k,r}(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ due to its product form,

$$\sum_{[n,m]} \psi_{k,r}(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}) = \sum_{\mathcal{P}} \prod_{V \in \mathcal{P}} (-1)^{|V|-1} \, (|V|-1)! \, S(V), \tag{19}$$

where $\sum_{[n,m]}$ denotes summation over all the permutations of $(i_1, \ldots, i_m)$ a subset of $\{1, \ldots, n\}$, $\sum_{\mathcal{P}}$ denotes the sum over all possible partitions $\mathcal{P}$ of $\{1, \ldots, m\}$, and $S(V) := \sum_{i=1}^{n} (g_{k,r}(\mathbf{X}_i))^{|V|}$.

Consider a partition $\mathcal{P}$ having j subsets, for which j_1 of them are of size 1, j_2 of them are of size 2, …, and j_m of them of size m. Then, $\sum_{\ell=1}^{m} j_\ell = j$ and $\sum_{\ell=1}^{m} \ell \, j_\ell = m$, and

$$(19) = \sum_{\mathcal{P}} \prod_{\ell=1}^{m} \left[C_\ell \left(\sum_{i=1}^{n} (g_{k,r}(\mathbf{X}_i))^\ell \right) \right]^{j_\ell}, \tag{20}$$

with $C_\ell := (-1)^{(\ell-1)} \, (\ell-1)!$.

Note that the finite U-statistic can be expressed as

$$\begin{aligned} U_{m,w,K}^{(n)} &= n^{m/2} \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k \psi_{k,r}(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}) \\ &= n^{m/2} \binom{n}{m}^{-1} \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k \sum_{1 \leq i_1 < \cdots < i_m \leq n} \psi_{k,r}(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}) =: \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k W_{m,k,r}^{(n)}, \end{aligned}$$

which by the kernel's symmetry and (20),

$$\begin{aligned} W_{m,k,r}^{(n)} &= n^{m/2} \binom{n}{m}^{-1} (m!)^{-1} \sum_{[n,m]} \psi_{k,r}(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}) \\ &= A_{n,m} \sum_{\mathcal{P}} n^{-m/2} \prod_{\ell=1}^{m} \left[C_\ell \left(\sum_{i=1}^{n} (g_{k,r}(\mathbf{X}_i))^\ell \right) \right]^{j_\ell} \\ &= A_{n,m} \sum_{\mathcal{P}} \left\{ n_{\mathcal{P}} \left(\prod_{\ell=1}^{m} C_\ell^{j_\ell} \right) \left(n^{-1/2} \sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i) \right)^{j_1} \prod_{\ell=2}^{m} \left(n^{-1} \sum_{i=1}^{n} (g_{k,r}(\mathbf{X}_i))^\ell \right)^{j_\ell} \right\}, \end{aligned}$$

with $A_{n,m} := n^m \dfrac{(n-m)!}{n!}$ and $n_{\mathcal{P}} := n^{-(m - j_1 - 2j_2 - \cdots - 2j_m)/2}$.

Now, we study $\mathbf{W}_m^{(n)} := (\mathbf{W}_{m,1}^{(n)\prime}, \ldots, \mathbf{W}_{m,K}^{(n)\prime})'$, with $\mathbf{W}_{m,k}^{(n)} := (W_{m,k,1}^{(n)}, \ldots, W_{m,k,d_{q,k}}^{(n)})'$. In the following, we denote $\mathbf{a} \odot \mathbf{b} = (a_1 b_1, \ldots, a_D b_D)$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^D$ with $D := \sum_{k=1}^{K} d_{q,k}$. Thus,

$$\begin{aligned} \mathbf{W}_m^{(n)} &= A_{n,m} \sum_{\mathcal{P}} \left\{ n_{\mathcal{P}} \Big(\prod_{\ell=1}^{m} C_\ell^{j_\ell} \Big) \left(\sqrt{n} \mathbf{T}_{n,1} \right)^{j_1} \odot \prod_{\ell=2}^{m} \mathbf{T}_{n,\ell}^{j_\ell} \right\} \\ &= A_{n,m} \left[\sum_{\mathcal{P}_{1,2}} \left\{ (-1)^{j_2} \left(\sqrt{n} \mathbf{T}_{n,1} \right)^{j_1} \odot \mathbf{T}_{n,2}^{j_2} \right\} \right. \\ &\qquad \left. + \sum_{\mathcal{P}_{>2}} \left\{ n_{\mathcal{P}_{>2}} \Big(\prod_{\ell=1}^{m} C_\ell^{j_\ell} \Big) \left(\sqrt{n} \mathbf{T}_{n,1} \right)^{j_1} \odot \prod_{\ell=2}^{m} \mathbf{T}_{n,\ell}^{j_\ell} \right\} \right], \end{aligned} \tag{21}$$

where $\mathbf{T}_{n,\ell} := n^{-1}\sum_{i=1}^n \mathbf{g}(\mathbf{X}_i)^\ell$ with $\mathbf{g}$ defined in Lemma B.2, $\sum_{\mathcal{P}_{1,2}}$ denotes the sum over all the partitions consisting solely of elements of size one and/or two, and $\sum_{\mathcal{P}_{>2}}$ over the rest of partitions. In the second equality, note also that $n_{\mathcal{P}_{1,2}} = 1$. For each partition $\mathcal{P}_{1,2}$, we can compute the number of partitions that have p pairs and $m-2p$ singletons, which allows writing

$$(21) = A_{n,m}\Bigg[\sum_{p=0}^{\lfloor m/2\rfloor}\left\{\frac{(-1)^p\, m!}{2^p\,(m-2p)!\,p!}\left(\sqrt{n}\mathbf{T}_{n,1}\right)^{m-2p}\odot \mathbf{T}_{n,2}^p\right\} + \sum_{\mathcal{P}_{>2}}\left\{n_{\mathcal{P}_{>2}}\Big(\prod_{\ell=1}^m C_\ell^{j_\ell}\Big)\left(\sqrt{n}\mathbf{T}_{n,1}\right)^{j_1}\odot\prod_{\ell=2}^m \mathbf{T}_{n,\ell}^{j_\ell}\right\}\Bigg].$$

By Lemma B.2, $\sqrt{n}\mathbf{T}_{n,1} \rightsquigarrow \mathcal{N}_D(\mathbf{0},\mathbf{I}_D)$, and by WLLN and orthonormality of spherical harmonics, $\mathbf{T}_{n,2} \overset{\mathrm{P}}{\to} \mathbf{1}_D$. Then, by Slutsky's Theorem, we have that the first sum in the right hand side converges in distribution to $H_m(\mathbf{Z})$, with $\mathbf{Z}\sim\mathcal{N}_D(\mathbf{0},\mathbf{I}_D)$, by continuity of the transformation applied to both vectors. Analogously, the second sum converges in probability to $\mathbf{0}_D$, since $m > j_1 + 2j_2 + \cdots + 2j_m$ causing $n_{\mathcal{P}>2} \to 0$. Finally, consider the continuous function $f:\mathbb{R}^D\to\mathbb{R}$, such that $f(\boldsymbol{x}) := \sum_{k=1}^K\sum_{r=1}^{d_{q,k}} w_k x_{d(k,r)}$ with $d(k,r) := \sum_{j=1}^{k-1} d_{q,j} + r$. Then, for the statistic $U_{m,w,K}^{(n)} = f(\mathbf{W}_m^{(n)})$ we apply the continuous mapping theorem, and part (i) follows.

The proof of part (ii) follows from Lemma B.3 noting that, for any $k\geq 1$ and $1\leq r\leq d_{q,k}$, $w_{\mathbf{k},\mathbf{r}} = 0$ for all $\mathbf{k}\neq(k,\ldots,k)$ or $\mathbf{r}\neq(r,\ldots,r)$, and since $n_{\mathbf{k},\mathbf{r}}(k,r) = m$ for $\mathbf{k}=(k,\ldots,k)$ and $\mathbf{r}=(r,\ldots,r)$. □

Proof of Theorem 4.1. Let $\phi_{m,w,K}^{(n)}$, $\phi_{m,w}^{(n)}$, $\phi_{m,w,K}^{\infty}$, and $\phi_{m,w}^{\infty}$ denote the characteristic functions of $U_{m,w,K}^{(n)}$, $U_{m,w}^{(n)}$, $U_{m,w,K}^{\infty}$, and $U_{m,w}^{\infty}$, respectively, and let $W_{m,k,r}$ the U-statistic associated to $\psi_{k,r}$ (see definition in Lemma B.4). The idea of the proof is to show that $\phi_{m,w}^{(n)}(t)\to\phi_{m,w}^{\infty}(t)$ for all $t\in\mathbb{R}$ by showing that for all $\epsilon>0$

$$\begin{aligned}\left|\phi_{m,w}^{(n)}(t)-\phi_{m,w}^{\infty}(t)\right| \leq{}& \left|\phi_{m,w}^{(n)}(t)-\phi_{m,w,K}^{(n)}(t)\right| + \left|\phi_{m,w,K}^{(n)}(t)-\phi_{m,w,K}^{\infty}(t)\right|\\ &+\left|\phi_{m,w,K}^{\infty}(t)-\phi_{m,w}^{\infty}(t)\right| < \epsilon,\end{aligned}\tag{22}$$

for all $n\geq N$ by choosing an N large enough to make the right-hand side terms arbitrarily small.

Let $\epsilon>0$. We first show that the sequence $\left(U_{m,w,K}^{(n)}\right)_{K=1}^{\infty}$ is a uniformly (on n) Cauchy sequence in the Hilbert space $L^2(\Omega) := \{X:\Omega\to\mathbb{R} : \int X(\omega)^2\,\mathrm{dP}(\omega) < \infty\}$, where $(\Omega,\mathcal{A},\mathrm{P})$ denotes the common probability space of the random variables $\mathbf{X}_1,\ldots,\mathbf{X}_n$. Let $L_2 > L_1 \geq 1$, then,

$$\begin{aligned}\mathrm{E}\Big[\big(U_{m,w,L_2}^{(n)} - U_{m,w,L_1}^{(n)}\big)^2\Big] &= \mathrm{E}\left[\left(\sum_{k=L_1+1}^{L_2}\sum_{r=1}^{d_{q,k}} w_k W_{m,k,r}^{(n)}\right)^2\right]\\ &= \sum_{k,\ell=L_1+1}^{L_2}\sum_{r=1}^{d_{q,k}}\sum_{s=1}^{d_{q,\ell}} w_k\, w_\ell\,\mathrm{E}\left[W_{m,k,r}^{(n)}\,W_{m,\ell,s}^{(n)}\right]\\ &= A_{n,m}\sum_{k,\ell=L_1+1}^{L_2}\sum_{r=1}^{d_{q,k}}\sum_{s=1}^{d_{q,\ell}} w_k\,w_\ell\,\delta_{k\ell}\,\delta_{rs}\\ &= A_{n,m}\sum_{k=L_1+1}^{L_2} w_k^2\, d_{q,k},\end{aligned}$$

with $A_{n,m} := n^m\binom{n}{m}^{-1}$, where in the third equality we used Lemma B.4. By the square summability of w, there exists some $K\geq 1$ such that for all $L_1, L_2 \geq K$, $\mathrm{E}\big[\big(U_{m,w,L_2}^{(m)} - U_{m,w,L_1}^{(m)}\big)^2\big] < \epsilon/3$.

Note that

$$\frac{\partial}{\partial n}\left(A_{n,m}\right)=n^{m-1}\frac{\Gamma\left(m+1\right)\Gamma\left(n+1-m\right)}{\Gamma\left(n+1\right)}\left(m+n\left(\psi^{(0)}(n-m+1)-\psi^{(0)}\left(n+1\right)\right)\right),$$

which, by the strict monotonicity of the digamma function, $\psi^{(0)}(x-m)-\psi^{(0)}(x)<0$ for all $m\geq 2$ and $x>m$, and therefore, $A_{n+1,m}<A_{n,m}$ for all $n>m$. Thus, $\mathrm{E}\big[\big(U^{(n)}_{m,w,L_2}-U^{(n)}_{m,w,L_1}\big)^2\big]<\epsilon/3$ for all $n>m$. By completeness of $L^2(\Omega)$, $\big(U^{(n)}_{m,w,K}\big)_{K=1}^{\infty}$ converges to some unique limit, $U^{(n)}_{m,w}$, in $L^2(\Omega)$ uniformly on $n>m$. Finally, since

$$\begin{aligned}\left|\phi^{(n)}_{m,w}(t)-\phi^{(n)}_{m,w,K}(t)\right| &= \left|\mathrm{E}\left[e^{itU^{(n)}_{m,w}}-e^{itU^{(n)}_{m,w,K}}\right]\right| \leq \mathrm{E}\left[\left|e^{it\left(U^{(n)}_{m,w}-U^{(n)}_{m,w,K}\right)}-1\right|\right]\\ &\leq \mathrm{E}\left[|t|\left|U^{(n)}_{m,w}-U^{(n)}_{m,w,K}\right|\right]\leq |t|\left(\mathrm{E}\left[\left(U^{(n)}_{m,w}-U^{(n)}_{m,w,K}\right)^2\right]\right)^{1/2},\end{aligned}$$

there exists a $K_0\geq 1$ such that, for all $K\geq K_0$, $\left|\phi^{(n)}_{m,w}(t)-\phi^{(n)}_{m,w,K}(t)\right|<\epsilon/3$ uniformly for all $n>m$.

Secondly, we prove the sequence $\big(U^{\infty}_{m,w,K}\big)_{K=1}^{\infty}$ is Cauchy in $L^2(\Omega)$. By Lemma B.5, we have

$$\begin{aligned}\mathrm{E}\left[\left(U^{\infty}_{m,w,L_2}-U^{\infty}_{m,w,L_1}\right)^2\right] &= \mathrm{E}\left[\sum_{k,\ell=L_1+1}^{L_2}\sum_{r=1}^{d_{q,k}}\sum_{s=1}^{d_{q,\ell}} w_k\, w_\ell\, H_m(Z_{k,r})\, H_m(Z_{\ell,s})\right]\\ &= \sum_{k=L_1+1}^{L_2}\sum_{r=1}^{d_{q,k}} w_k^2\mathrm{E}\left[\,H_m^2(Z_{k,r})\right]\\ &= m!\sum_{k=L_1+1}^{L_2} w_k^2\, d_{q,k}.\end{aligned}$$

Then, by square summability assumption of w, there exists a $K\geq 1$ such that for all $L_1,L_2\geq K$, $\mathrm{E}\big[\big(U^{\infty}_{m,w,L_2}-U^{\infty}_{m,w,L_1}\big)^2\big]<\epsilon/3$. By completeness of $L^2(\Omega)$, $U^{\infty}_{m,w,K}$ converges in $L^2(\Omega)$ to some unique limit, $U^{\infty}_{m,w}$. Convergence in mean square implies convergence in distribution, and therefore the convergence of the corresponding characteristic functions by the continuity theorem. Thus, there exists a $K_1\geq 1$ such that, for all $K\geq K_1$, $\left|\phi^{\infty}_{m,w,K}(t)-\phi^{\infty}_{m,w}(t)\right|<\epsilon/3$.

Thirdly, for a given $K\geq 1$, by Proposition 4.1(i) and the continuity theorem of characteristic functions, there exists an $N\geq 1$, such that for all $n\geq N$, $\left|\phi^{(n)}_{m,w,K}(t)-\phi^{\infty}_{m,w,K}(t)\right|<\epsilon/3$.

Therefore, choosing a $K\geq\max\{K_0,K_1\}$, for all $n\geq\max\{m+1,N\}$, $\left|\phi^{(n)}_{m,w}(t)-\phi^{\infty}_{m,w}(t)\right|<\epsilon$ in (22), and (10) follows. □

Proof of Theorem 4.2. Since (12) holds by assumption, $V^{(n)}_{m,w}$ can be expressed in the form of (11). Now, consider m even. Let $\mathcal{L}^{(n)}_{m,w,K}$, $\mathcal{L}^{(n)}_{m,w}$, $\mathcal{L}^{\infty}_{m,w,K}$, and $\mathcal{L}^{\infty}_{m,w}$ be the distribution of $V^{(n)}_{m,w,K}$, $V^{(n)}_{m,w}$, $V^{\infty}_{m,w,K}$, and $V^{\infty}_{m,w}$, respectively. Following Definition 6.8 together with Theorem 6.9 in Villani (2009), for proving weak convergence $\mathcal{L}^{(n)}_{m,w}\to\mathcal{L}^{\infty}_{m,w}$ it suffices to show $W_p\big(\mathcal{L}^{(n)}_{m,w},\mathcal{L}^{\infty}_{m,w}\big)\to 0$ as $n\to\infty$ for some $p\geq 1$, where $W_p\left(\mu,\nu\right)$ is the Wasserstein distance of order p between distributions μ and ν, and is given by

$$W_p(\mu,\nu)=\left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}}|x-y|^p\,\mathrm{d}\pi(x,y)\right)^{1/p},$$

where $\Pi(\mu,\nu)$ is the set of all couplings between ν and μ.

The proof relies on

$$\begin{aligned}W_p\big(\mathcal{L}^{(n)}_{m,w},\mathcal{L}^{\infty}_{m,w}\big)\leq{}& W_p\big(\mathcal{L}^{(n)}_{m,w},\mathcal{L}^{(n)}_{m,w,K}\big)+W_p\big(\mathcal{L}^{(n)}_{m,w,K},\mathcal{L}^{\infty}_{m,w,K}\big)\\ &+W_p\left(\mathcal{L}^{\infty}_{m,w,K},\mathcal{L}^{\infty}_{m,w}\right),\end{aligned}\tag{23}$$

and proving that we can choose an N large enough so that the three right-hand terms are arbitrarily close to zero.

Let $\epsilon > 0$. We first show that for a sufficiently large N_0, there exists a K_0 such that for all $K > K_0$, $\sup_{n \geq N_0} W_1\big(\mathcal{L}^{(n)}_{m,w}, \mathcal{L}^{(n)}_{m,w,K}\big) < \epsilon/3$. Using the specific coupling π induced by the data generation process, we readily have

$$\sup_{n \geq N_0} W_1\big(\mathcal{L}^{(n)}_{m,w}, \mathcal{L}^{(n)}_{m,w,K}\big) \leq \sup_{n \geq N_0} \mathrm{E}\big[|V^{(n)}_{m,w} - V^{(n)}_{m,w,K}|\big].$$

Using the monotone convergence theorem since m is even and w is nonnegative,

$$\begin{aligned}
\mathrm{E}\big[|V^{(n)}_{m,w} - V^{(n)}_{m,w,K}|\big] &= \mathrm{E}\left[\sum_{k=K+1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right] \\
&= \sum_{k=K+1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right]. \qquad (24)
\end{aligned}$$

By Lemma B.7, there exists an $N_0 > m$ independent of k such that for all $n \geq N_0$,

$$(24) \leq 2 \sum_{k=K+1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \bigg(\frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A} \\ c_{\boldsymbol{\alpha}} < m/2}} n^{c_{\boldsymbol{\alpha}} - m/2 + 1}\, C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i}\bigg).$$

Using Lemma 4.1, among all possible configurations $\boldsymbol{\alpha} \in \mathcal{A}$, since $\lambda_q(a+b) > \lambda_q(a) + \lambda_q(b)$ for all integers $a, b > 1$ and every dimension $q \geq 1$, the one that constrains the bound is $\boldsymbol{\alpha}^* = (0, \ldots, 0, m)$, hence for any $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}^*$, $\prod_{i=1}^{n} e_{k,r,\alpha_i} \leq A_m k^{\lambda_q(m)}$, $A_m \in \mathbb{R}$, for all $k \geq K^*$, K^* being sufficiently large, and we obtain

$$\begin{aligned}
&2 \sum_{k=K^*+1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \bigg(\frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A} \\ c_{\boldsymbol{\alpha}} < m/2}} n^{c_{\boldsymbol{\alpha}} - m/2 + 1}\, C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i}\bigg) \\
&\quad \leq 2 \sum_{k=K^*+1}^{\infty} w_k \sum_{r=1}^{d_{q,k}} \bigg(\frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} A_m \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A} \\ c_{\boldsymbol{\alpha}} < m/2}} n^{c_{\boldsymbol{\alpha}} - m/2 + 1}\, C_{\boldsymbol{\alpha}}\, k^{\lambda_q(m)}\bigg) \\
&\quad = 2 \sum_{k=K^*+1}^{\infty} w_k\, d_{q,k} \left(\frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} A_m D_{n,m}\, k^{\lambda_q(m)}\right) \\
&\quad = \frac{2\, m!}{(m/2)!\, 2^{m/2}} \sum_{k=K^*+1}^{\infty} w_k\, d_{q,k} + 2n^{-1} A_m D_{n,m} \sum_{k=K^*+1}^{\infty} w_k\, d_{q,k} k^{\lambda_q(m)} \\
&\quad \leq \frac{2\, m!}{(m/2)!\, 2^{m/2}} \sum_{k=K^*+1}^{\infty} w_k\, d_{q,k} + 4n^{-1} A_m D_{n,m} \sum_{k=K^*+1}^{\infty} w_k\, k^{(q-1)+\lambda_q(m)},
\end{aligned}$$

with $D_{n,m} := \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A} \\ c_{\boldsymbol{\alpha}} < m/2}} n^{c_{\boldsymbol{\alpha}} - m/2 + 1}\, C_{\boldsymbol{\alpha}} = O(1)$, where in the last inequality we used $d_{q,k} \sim k^{q-1}$, with $\sim$ denoting asymptotic equivalence.

Since $\sum_{k=1}^{\infty} w_k\, d_{q,k}^{(m+1)/2} < \infty$ implies $\sum_{k=1}^{\infty} w_k\, d_{q,k} < \infty$ for $m > 2$, then there exists a K_0 such that for all $K \geq K_0$, $\mathrm{E}\big[|V^{(n)}_{m,w} - V^{(n)}_{m,w,K}|\big] < \epsilon/3$ for all $n \geq N_0$, since the second right-hand side sum is scaled by n^{-1}.

Secondly, Theorem 6.9 in Villani (2009) states that $W_1\big(\mathcal{L}^{(n)}_{m,w,K}, \mathcal{L}^{\infty}_{m,w,K}\big) \to 0$ if $\mathcal{L}^{(n)}_{m,w,K}$ converges weakly to $\mathcal{L}^{\infty}_{m,w,K}$ in the Wasserstein space of order one, which, according to Definition 6.8 in ibid, occurs if $\mathcal{L}^{(n)}_{m,w,K} \to \mathcal{L}^{\infty}_{m,w,K}$ and $\mathrm{E}\big[|V^{(n)}_{m,w,K}|\big] \to \mathrm{E}\big[|V^{\infty}_{m,w,K}|\big]$. For a given K, weak convergence of distributions holds from Proposition 4.1(ii) while the second condition, since m is even,

$$\mathrm{E}\big[|V^{(n)}_{m,w,K}|\big] = \mathrm{E}\big[V^{(n)}_{m,w,K}\big] = \mathrm{E}\left[n^{-m/2}\sum_{k=1}^{K}\sum_{r=1}^{d_{q,k}} w_k\left(\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right]$$
$$= \sum_{k=1}^{K} w_k \sum_{r=1}^{d_{q,k}} \mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right],$$

and using Lemma B.7,

$$\mathrm{E}\big[|V^{(n)}_{m,w,K}|\big] \to \sum_{k=1}^{K} w_k d_{q,k}\,\frac{m!}{2^{m/2}\,(m/2)!} = \mathrm{E}\left[|V^{\infty}_{m,w,K}|\right]$$

holds. Consequently, there is an N_1 such that for all $n \geq N_1$,

$$W_1\big(\mathcal{L}^{(n)}_{m,w,K}, \mathcal{L}^{\infty}_{m,w,K}\big) < \frac{\epsilon}{3}.$$

Thirdly, we consider the specific coupling π between $\mathcal{L}^{\infty}_{m,w}$ and $\mathcal{L}^{\infty}_{m,w,K}$ such that the set of iid normal random variables $\{Z_{k,r}\}$ is common for both distributions, that is, $\mathcal{L}^{\infty}_{m,w,K} \overset{d}{=} \sum_{k=1}^{K}\sum_{r=1}^{d_{q,k}} w_k Z^m_{k,r}$, and $\mathcal{L}^{\infty}_{m,w} \overset{d}{=} \sum_{k=1}^{K}\sum_{r=1}^{d_{q,k}} w_k Z^m_{k,r} + \sum_{k=K+1}^{\infty}\sum_{r=1}^{d_{q,k}} w_k Z^m_{k,r}$. We have that, because m is even and from the summability condition (12),

$$W_1\big(\mathcal{L}^{\infty}_{m,w,K}, \mathcal{L}^{\infty}_{m,w}\big) \leq \mathrm{E}\left[\left|V^{\infty}_{m,w,K} - V^{\infty}_{m,w}\right|\right] = \mathrm{E}\left[\left|\sum_{k=K+1}^{\infty}\sum_{r=1}^{d_{q,k}} w_k Z^m_{k,r}\right|\right]$$
$$= \frac{m!}{2^{m/2}(m/2)!}\sum_{k=K+1}^{\infty} w_k\, d_{q,k} < \frac{\epsilon}{3}$$

for every $K \geq K_1$, with K_1 sufficiently large.

Thus, choosing a $K \geq \max\{K_0, K_1\}$, for all $n \geq \max\{N_0, N_1\}$, $W_1\big(\mathcal{L}^{(n)}_{m,w}, \mathcal{L}^{\infty}_{m,w}\big) < \epsilon$ in (23), and (13) follows. □

A.3 Proofs of Section 5

Proof of Proposition 5.1. Let $k \geq 1$, $1 \leq r \leq d_{q,k}$, and $T_{n,k,r,\ell} := n^{-1}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)^{\ell}$ for $\ell \geq 1$. Let K be any integer such that $K > \min\{k : (k,r) \in \mathcal{S}_{\neq}\}$.

For the V-statistic, since (12) is met, m is even, and $w_k > 0$ for all $k \geq 1$, we have

$$V^{(n)}_{m,w} \geq V^{(n)}_{m,w,K} = \sum_{k=1}^{K}\sum_{r=1}^{d_{q,k}} w_k(\sqrt{n}T_{n,k,r,1})^m$$
$$= \sum_{\substack{(k,r)\in\mathcal{S}_{\neq}\\ k\leq K}} w_k(\sqrt{n}T_{n,k,r,1})^m + \sum_{\substack{(k,r)\notin\mathcal{S}_{\neq}\\ k\leq K}} w_k(\sqrt{n}T_{n,k,r,1})^m =: A_{n,m,w,K} + B_{n,m,w,K}$$

Note that $A_{n,m,w,K} \overset{\mathrm{P}}{\to} +\infty$ since each term of the finite sum $(\sqrt{n}T_{n,k,r,1})^m \overset{\mathrm{P}}{\to} +\infty$, due to Lemma B.9 cases (i) and (iii), and since m is even. For $B_{n,m,w,K}$, since the associated coefficients $h_{k,r} = 0$, by Lemma B.9 part (ii), $B_{n,m,w,K} = O_{\mathrm{P}}(1)$ since it is a finite sum.

Therefore, $V^{(n)}_{m,w} \geq V^{(n)}_{m,w,K} \overset{\mathrm{P}}{\to} +\infty$, and cases (i) and (ii) are proven for V-statistics.

For U-statistics, we first show that if $h_{k,r} \neq 0$, $W_{m,k,r}^{(n)} \xrightarrow{\mathrm{P}} +\infty$, with $W_{m,k,r}^{(n)}$ as defined in the Proof of Proposition 4.1. From (21),

$$
\begin{aligned}
W_{m,k,r}^{(n)} &= A_{n,m}\Bigg[\sum_{p=0}^{\lfloor m/2\rfloor}\left\{\frac{(-1)^p\, m!}{2^p\ (m-2p)!\, p!}\left(\sqrt{n}T_{n,k,r,1}\right)^{m-2p} T_{n,k,r,2}^{p}\right\} \\
&\quad + \sum_{\mathcal{P}_{>2}}\left\{n^{j_1/2} n_{\mathcal{P}_{>2}}\left(T_{n,k,r,1}\right)^{j_1}\prod_{\ell=2}^{m}\left(C_\ell T_{n,k,r,\ell}\right)^{j_\ell}\right\}\Bigg] \\
&= A_{n,m}\Bigg[\left(\sqrt{n}T_{n,k,r,1}\right)^{m} + \sum_{p=1}^{\lfloor m/2\rfloor}\left\{\frac{(-1)^p\, m!}{2^p\ (m-2p)!\, p!}\left(\sqrt{n}T_{n,k,r,1}\right)^{m-2p} T_{n,k,r,2}^{p}\right\} \\
&\quad + \sum_{\mathcal{P}_{>2}}\left\{n^{j-m/2}\left(T_{n,k,r,1}\right)^{j_1}\prod_{\ell=2}^{m}\left(C_\ell T_{n,k,r,\ell}\right)^{j_\ell}\right\}\Bigg] \\
&=: A_{n,m}\left(C_{n,m,k,r} + D_{n,m,k,r} + E_{n,m,k,r}\right),
\end{aligned}
$$

where in the second equality $j = \sum_{\ell=1}^{m} j_\ell$ denotes the number of elements in the corresponding partition $\mathcal{P}_{>2}$. Now, note that $n^{j_1/2} n_{\mathcal{P}_{>2}} = n^{-m/2+j_1+j_2+\cdots+j_m} = n^{j-m/2} = O(n^{m/2-1})$, since $j < m$, because the only partition with m elements is the one with all singletons. Also, $T_{n,k,r,\ell} \xrightarrow{\mathrm{P}} e_{k,r,\ell}$ with $e_{k,r,\ell} := \mathrm{E}_H\left[g_{k,r}^{\ell}(\mathbf{X}_1)\right]$. Thus, $E_{n,m,k,r} = O_{\mathrm{P}}(n^{m/2-1})$ and $D_{n,m,k,r} = O_{\mathrm{P}}(n^{m/2-1})$. For the first term, note that m is even, and using Lemma B.9 parts (i) and (iii), we can conclude that $C_{n,m,k,r} \xrightarrow{\mathrm{P}} +\infty$ and $C_{n,m,k,r} = O_{\mathrm{P}}(n^{m/2})$. Recall that $A_{n,m} = n^m(n-m)!/n! \to 1$, therefore, $W_{m,k,r}^{(n)} \xrightarrow{\mathrm{P}} +\infty$.

Now, since $w_k > 0$ for all $k \geq 1$, and with the assumption that $\mathcal{S}_{\neq}$ is finite, we have

$$
\begin{aligned}
U_{m,w}^{(n)} &= n^{m/2}\binom{n}{m}^{-1}\sum_{1\leq i_1<\cdots<i_m\leq n}\ \sum_{k=1}^{\infty}\sum_{r=1}^{d_{q,k}} w_k \psi_{k,r}(\mathbf{X}_{i_1},\ldots,\mathbf{X}_{i_m}) \\
&= \sum_{(k,r)\in\mathcal{S}_{\neq}} w_k W_{k,r}^{(n)} + \sum_{(k,r)\notin\mathcal{S}_{\neq}} w_k W_{k,r}^{(n)}.
\end{aligned}
\tag{25}
$$

The first term diverges to $+\infty$ in probability since it is a finite sum. The second term is $O_{\mathrm{P}}(1)$, since the associated coefficients $h_{k,r} = 0$, and, thus, it can be shown to converge in distribution given the square summability of coefficients and case (ii) of Lemma B.9, following an analogous proof as in Theorem 4.1. Therefore, $U_{m,w}^{(n)} \xrightarrow{\mathrm{P}} +\infty$, and (iii) is proven. For (i), both sums in (25) are finite, where the second term converges in distribution following an analogous proof as in Proposition 4.1(i), and the result for finite U-statistics is proven. □

Remark A.1. *The behavior of $U_{m,w}^{(n)}$ under an alternative H such that $\mathcal{S}_{\neq}$ is not finite requires additional technical assumptions to (i) allow the rearrangement in* (25)*, in particular, absolute summability is required, for which condition* (12) *on the weights w would suffice; and (ii) to ensure some terms in the first series of* (25) *are controlled, since the infinite sum of terms diverging in probability to $+\infty$ is not guaranteed to diverge in probability to $+\infty$.*

Proof of Proposition 5.2. The proof of part (i) follows from Lemma B.10 and applying the continuous mapping theorem as in Proposition 4.1(i). The proof of part (ii) follows from Lemma B.10, applying the continuous mapping theorem as in Lemma B.3, and noting that, for any $k \geq 1$ and $1 \leq r \leq d_{q,k}$, $w_{\mathbf{k},\mathbf{r}} = 0$ for all $\mathbf{k} \neq (k,\ldots,k)$ or $\mathbf{r} \neq (r,\ldots,r)$, and since $n_{\mathbf{k},\mathbf{r}}(k,r) = m$ for $\mathbf{k} = (k,\ldots,k)$ and $\mathbf{r} = (r,\ldots,r)$. □

Proof of Proposition 5.3. In this proof, we put $\mathrm{Supp}(w) := \{k : w_k \neq 0\}$. First, consider part (i). Following the proof of Proposition 4.1(i), we have that

$$U_{m,w,K}^{(n)} = \sum_{k=1}^{K} \sum_{r=1}^{d_{q,k}} w_k W_{m,k,r}^{(n)} = \sum_{k=k_w}^{K} \sum_{r=1}^{d_{q,k}} w_k W_{m,k,r}^{(n)}$$

which by the kernel symmetry and (20),

$$W_{m,k,r}^{(n)} = A_{n,m} \sum_{\mathcal{P}} \left\{ n_{\mathcal{P}} \left(\prod_{\ell=1}^{m} C_\ell^{j_\ell} \right) \left(n^{-1/2} \sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i) \right)^{j_1} \prod_{\ell=2}^{m} \left(n^{-1} \sum_{i=1}^{n} \left(g_{k,r}(\mathbf{X}_i) \right)^{\ell} \right)^{j_\ell} \right\}.$$

Now, still as in the proof of Proposition 4.1(i), we have

$$\begin{aligned} \mathbf{W}_m^{(n)} = A_{n,m} \Bigg[&\sum_{\mathcal{P}_{1,2}} \left\{ (-1)^{j_2} \left(\sqrt{n} \mathbf{T}_{n,1} \right)^{j_1} \odot \mathbf{T}_{n,2}^{j_2} \right\} \\ &+ \sum_{\mathcal{P}_{>2}} \left\{ n_{\mathcal{P}_{>2}} \left(\prod_{\ell=1}^{m} C_\ell^{j_\ell} \right) \left(\sqrt{n} \mathbf{T}_{n,1} \right)^{j_1} \odot \prod_{\ell=2}^{m} \mathbf{T}_{n,\ell}^{j_\ell} \right\} \Bigg]. \end{aligned} \tag{26}$$

Decomposing

$$\sqrt{n} \mathbf{T}_{n,1} = \sqrt{n} (\mathbf{T}_{n,1} - \mathrm{E}[\mathbf{T}_{n,1}]) + \sqrt{n} \mathrm{E}[\mathbf{T}_{n,1}]$$

it follows directly from Proposition 5 in García-Portugués et al (2025) that $\sqrt{n}(\mathbf{T}_{n,1} - \mathrm{E}[\mathbf{T}_{n,1}])$ is asymptotically standard normal under $\mathrm{P}_{\kappa_n,f}^{(n)}$ (which also holds under an arbitrary location $\boldsymbol{\mu} \in \mathbb{S}^q$ different than $\mathbf{e}_1 := (1, 0, \ldots, 0)'$). It only remains to prove that under $\mathrm{P}_{\kappa_n,f}^{(n)}$,

$$\sqrt{n} \mathrm{E}[\mathbf{g}_k(\mathbf{X}_1)] = \frac{m_{k,k_*} \, f^{\underline{k_*}}(0) \, \tau^{k_*}}{a_{q,k} \, (k_*!)} \mathbf{g}_k(\boldsymbol{\mu}) \, \mathbb{1}_{\{k \sim k_*, k_\dagger \leq k \leq k_*\}} + o(1). \tag{27}$$

This follows from Lemma B.11 together with Proposition 4 in García-Portugués et al (2025), which also holds for an arbitrary $\boldsymbol{\mu} \in \mathbb{S}^q$, and noting that $a_{q,k} = h_{q,k}(1) \, t_{q,k}^2$, with $t_{q,k} = \sqrt{2} \cdot \mathbb{1}_{\{q=1\}} + (1 + 2k/(q-1)) / \sqrt{d_{q,k}} \cdot \mathbb{1}_{\{q>1\}}$.

It follows that $\sqrt{n} \mathbf{T}_{n,1}$ is $O_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}_{\kappa_n,f}^{(n)}$. As a result, the second term in (26) is $o_{\mathrm{P}}(1)$, and (i) directly follows from (27) and the continuous mapping theorem as in the proof of Proposition 4.1(i).

The proof of part (ii) follows the same decomposition of $\sqrt{n} \mathbf{T}_{n,1}$ as in the proof of part (i), and applies the continuous mapping theorem as in Lemma B.3, noting that, for any $k \geq 1$ and $1 \leq r \leq d_{q,k}$, $w_{\mathbf{k},\mathbf{r}} = 0$ for all $\mathbf{k} \neq (k, \ldots, k)$ or $\mathbf{r} \neq (r, \ldots, r)$, and that $n_{\mathbf{k},\mathbf{r}}(k, r) = m$ for $\mathbf{k} = (k, \ldots, k)$ and $\mathbf{r} = (r, \ldots, r)$. □

A.4 Proofs of Section 7

Proof of Lemma 7.1. Note that in the case $q = 1$, $\Phi_{m,\delta_\ell}(\boldsymbol{\theta}) = \sum_{r=1}^{d_{q,\ell}} \psi_{\ell,r}(\mathbf{X}_1, \ldots, \mathbf{X}_m)$, and consider it in the form obtained in Lemma B.1. Since a rotated point, $\mathbf{Ox}$, in $\mathbb{S}^1$ can be represented in polar coordinates as $\theta + \alpha$, with $\alpha \in (0, 2\pi]$, the first equality holds. Let $I^m := \{-1, 1\}^m$. For m odd,

$$\begin{aligned} \int_0^{2\pi} \Phi_{m,\delta_\ell}(\boldsymbol{\theta} + \boldsymbol{\alpha}) \, \mathrm{d}\alpha &= 2^{(1-m)/2} \sum_{\mathbf{e} \in I^m} \int_0^{2\pi} \cos\left(\ell \mathbf{e}'(\boldsymbol{\theta} + \boldsymbol{\alpha}) - \mu_{\mathbf{e}} \right) \, \mathrm{d}\alpha \\ &= 2^{(1-m)/2} \sum_{\mathbf{e} \in I^m} \int_0^{2\pi} \cos\left(\ell s_{\mathbf{e}} \alpha + \ell \mathbf{e}' \boldsymbol{\theta} - \mu_{\mathbf{e}} \right) \, \mathrm{d}\alpha. \end{aligned} \tag{28}$$

Note that since m is odd, $s_{\mathbf{e}} \neq 0$ for all $\mathbf{e} \in I^m$, thus,

$$(28) = \frac{2^{(1-m)/2}}{\ell} \sum_{\mathbf{e} \in I^m} \frac{1}{s_{\mathbf{e}}} \left[\sin\left(\ell \alpha s_{\mathbf{e}} + \ell \mathbf{e}'\boldsymbol{\theta} - \mu_{\mathbf{e}}\right)\right]_0^{2\pi} = 0,$$

where last equality holds for all $\ell \in \mathbb{Z}$.

For m even, letting $\mathcal{S}_m := \{1\} \times I^{m-1}$, we have

$$\int_0^{2\pi} \Phi_{m,\delta_\ell}(\boldsymbol{\theta} + \boldsymbol{\alpha})\, \mathrm{d}\alpha = 2^{1-m/2} \sum_{\mathbf{e} \in \mathcal{S}_m} \int_0^{2\pi} \left(1 + (-1)^{m/2} p_{\mathbf{e}}\right) \cos\left(\ell \mathbf{e}'\boldsymbol{\theta} + \ell \alpha s_{\mathbf{e}}\right)\, \mathrm{d}\alpha. \tag{29}$$

In this case, there is some $\mathbf{e} \in \mathcal{S}_m$, such that $s_{\mathbf{e}} = 0$, hence,

$$(29) = 2^{1-m/2} \sum_{\substack{\mathbf{e} \in \mathcal{S}_m \\ s_{\mathbf{e}} = 0}} \int_0^{2\pi} \left(1 + (-1)^{m/2} p_{\mathbf{e}}\right) \cos\left(\ell \mathbf{e}'\boldsymbol{\theta}\right)\, \mathrm{d}\alpha.$$

Note that for $m \in 4\mathbb{Z}$, the leading term within the integral is 0 for all $\mathbf{e}$ such that $p_{\mathbf{e}} = -1$, and, analogously, when $p_{\mathbf{e}} = 1$ for $m \in 2\mathbb{Z} \setminus 4\mathbb{Z}$. However, noting that the sum is over any $\mathbf{e}$ such that $s_{\mathbf{e}} = 0$, this latter condition already implies the product condition for each case, and (18) follows. □

Proof of Proposition 7.1. The proof uses the following integrals for $k_1, \ell \geq 1$ integers, $\mathbf{e} := (e_1, \ldots, e_m)' \in \{-1, 1\}^m$, $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_m)' \in (0, 2\pi]^m$, and denoting $\mathbf{e}_{-1} := (e_2, \ldots, e_m)'$, and $\boldsymbol{\theta}_{-1} := (\theta_2, \ldots, \theta_m)'$,

$$\int_0^{2\pi} \cos\left(\ell \mathbf{e}'\boldsymbol{\theta}\right) \cos(k_1\theta_1)\, \mathrm{d}\theta_1 = \pi \cos\left(\ell \mathbf{e}_{-1}'\boldsymbol{\theta}_{-1}\right) \delta_{k_1 \ell} \tag{30}$$

$$\int_0^{2\pi} \sin\left(\ell \mathbf{e}'\boldsymbol{\theta}\right) \cos\left(k_1\theta_1\right)\, \mathrm{d}\theta_1 = \pi \sin\left(\ell \mathbf{e}_{-1}'\boldsymbol{\theta}_{-1}\right) \delta_{k_1 \ell} \tag{31}$$

$$\int_0^{2\pi} \cos\left(\ell \mathbf{e}'\boldsymbol{\theta}\right) \sin(k_1\theta_1)\, \mathrm{d}\theta_1 = -e_1 \pi \sin\left(\ell \mathbf{e}_{-1}'\boldsymbol{\theta}_{-1}\right) \delta_{k_1 \ell} \tag{32}$$

$$\int_0^{2\pi} \sin\left(\ell \mathbf{e}'\boldsymbol{\theta}\right) \sin\left(k_1\theta_1\right)\, \mathrm{d}\theta_1 = e_1 \pi \cos\left(\ell \mathbf{e}_{-1}'\boldsymbol{\theta}_{-1}\right) \delta_{k_1 \ell} \tag{33}$$

Letting $\tilde{g}_{k,1}(\theta) = \cos k\theta$ and $\tilde{g}_{k,2}(\theta) = \sin k\theta$, the basis coefficients are obtained by

$$\begin{aligned} u_{\mathbf{k},\mathbf{r}} &= (2\pi)^{-m} \int_{(\mathbb{S}^1)^m} \tilde{\Phi}_{m,\delta_\ell}(\boldsymbol{\theta})\, g_{k_1,r_1}(\theta_1) \cdots g_{k_m,r_m}(\theta_m)\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m \\ &= 2^{2-m} \pi^{-m} \sum_{\mathbf{e} \in \mathcal{S}_{m,0}} \int_{(\mathbb{S}^1)^m} \cos\left(\ell \mathbf{e}'\boldsymbol{\theta}\right) \tilde{g}_{k_1,r_1}(\theta_1) \cdots \tilde{g}_{k_m,r_m}(\theta_m)\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m. \end{aligned}$$

For $\mathbf{k} \neq (\ell, \ldots, \ell)$ and any $\mathbf{r}$, we have $u_{\mathbf{k},\mathbf{r}} = 0$ by orthogonality (30–33), and (i) follows.

On the one hand, for (ii), given S is odd, the basis function $\prod_{j=1}^m \tilde{g}_{k_j,r_j}$ consists of an odd number of $\sin(k_j\, \theta_j)$. Note that when iteratively integrating, a basis term of the form $\cos(k_j\, \theta_j)$ will result in the same trigonometric function type it is applied to, (30) and (31). In contrast, a term of $\sin(k_j\, \theta_j)$ will yield the cosine function if applied to sine and the other way around, (32) and (33). Therefore, an odd S will result in evaluating $\sin 0 = 0$.

On the other hand, if S is even, the basis function has an even number of $\sin(k_j\, \theta_j)$ terms, which results in evaluating $\cos 0 = 1$, and the integral is not null. For each pair of $\sin(k_j\, \theta_j)$ evaluated, a -1 term is added from (32), and the element e_j results after each integral involving a $\sin(k_j\, \theta_j)$ term. Thus, we have

$$u_{\mathbf{k},\mathbf{r}} = 2^{2-m}\, (-1)^{S/2} \sum_{\mathbf{e} \in \mathcal{S}_{m,0}} \prod_{j=1}^{S} e_{i_j},$$

with $(i_1, \ldots, i_S)$ being the index of the elements of $\mathbf{r}$ equal to 1.

Then, denoting the cardinality of a set A by $\#A$, we have for $S = m$, $u_{\mathbf{k},\mathbf{r}} = 2^{2-m}(-1)^{m/2} \sum_{\mathbf{e}\in\mathcal{S}_{m,0}} p_{\mathbf{e}} = 2^{2-m}(-1)^m \sum_{\mathbf{e}\in\mathcal{S}_{m,0}} 1 = 2^{2-m}\, \#\mathcal{S}_{m,0}$, since for every element $\mathbf{e} \in \mathcal{S}_{m,0}$, $p_{\mathbf{e}} = (-1)^{m/2}$, and for $S = 0$, we immediately obtain the same result. Note that $\#\mathcal{S}_{m,0} = \binom{m-1}{m/2}$.

For $0 < S < m$, $\prod_{j=1}^S e_{i_j}$ equals either $+1$ or -1. Let n_+ be the number of elements $\mathbf{e}$ in $\mathcal{S}_{m,0}$ such that $\prod_{j=1}^S e_{i_j} = 1$. Then, $\sum_{\mathbf{e}\in\mathcal{S}_{m,0}} \prod_{j=1}^S e_{i_j} = 2n_+ - \#\mathcal{S}_{m,0}$. Given that S is even, n_+ can be computed, first assuming $i_j \neq 1$ for all $1 \leq j \leq S$. In this case, an even number t of (-1) terms must be placed on S spots, and for each such combination, we place the $m/2 - t$ remaining (-1) elements in $m-1$ spots, therefore $n_+ = \sum_{t\in 2\mathbb{Z}} \binom{S}{t}\binom{m-1-S}{m/2-t}$. In the case of assuming $i_j = 1$ for some $1 \leq j \leq S$, an even number t of (-1) terms must be placed in $S-1$ spots, and for each of those combinations, the remaining $m/2 - t$ (-1) elements are placed on $m - S$ spots, thus $n_+ = \sum_{t\in 2\mathbb{Z}} \binom{S-1}{t}\binom{m-S}{m/2-t}$. These two expressions, $A := \sum_{t\in 2\mathbb{Z}} \binom{S}{t}\binom{m-1-S}{m/2-t}$ and $B := \sum_{t\in 2\mathbb{Z}} \binom{S-1}{t}\binom{m-S}{m/2-t}$ are equivalent considering S is even, the reason is that $2A$ and $2B$ both count the number of elements in $\{\mathbf{e} \in \{-1, 1\}^m : s_{\mathbf{e}} = 0\}$ such that $\prod_{j=1}^S e_{i_j} = 1$, where $0 \leq i_j \leq m$ for all $1 \leq j \leq S$ and $i_j \neq i_k$ for all $j \neq k$, and *(iii)* follows. □

Proof of Proposition 7.2. This result is a direct application of Lemma B.3. □

B Technical Lemmas

B.1 Lemmas of Section 3

The following lemma is required for proving Proposition 3.1. It finds an expression of a kernel with a sparse sequence of weights, $w_k = \delta_{k\ell}$ (non-zero only at $k = \ell$), which immediately includes Rayleigh ($\ell = 1$) and Bingham ($\ell = 2$) statistics.

Lemma B.1. *Let $m \geq 2$ and $\ell \geq 1$. Let $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_m)'$ be the polar angles of each element in $(\mathbf{X}_1, \ldots, \mathbf{X}_m)'$, and let $\Phi_{m,\delta_\ell}(\boldsymbol{\theta}) := \Phi_w(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ with $w = \{\delta_{k\ell}\}_{k=1}^\infty$. Then,*

$$\Phi_{m,\delta_\ell}(\boldsymbol{\theta}) = 2^{-m/2} \begin{cases} \sqrt{2} \sum_{\mathbf{e}\in I^m} \cos\left(\ell\mathbf{e}'\boldsymbol{\theta} - \mu_{\mathbf{e}}\right), & m \text{ odd}, \\ 2\sum_{\mathbf{e}\in\{1\}\times I^{m-1}} \left(1 + (-1)^{m/2} p_{\mathbf{e}}\right) \cos\left(\ell\mathbf{e}'\boldsymbol{\theta}\right), & m \text{ even}, \end{cases} \tag{34}$$

where $I^m := \{-1, 1\}^m$, $p_{\mathbf{e}} := \prod_{i=1}^m e_i$ with $\mathbf{e} = (e_1, \ldots, e_m)'$, and $\mu_{\mathbf{e}} := \tan^{-1}\left((-1)^{\lfloor m/2\rfloor} p_{\mathbf{e}}\right)$.

Proof of Lemma B.1. Notice that in $\mathbb{S}^1$, $g_{k,1}(\mathbf{x}) = \sqrt{2}\cos(k\theta)$ and $g_{k,2}(\mathbf{x}) = \sqrt{2}\sin(k\theta)$ for $k \geq 1$. Using the product-to-sum trigonometric identities,

$$\prod_{j=1}^m \cos\theta_j = 2^{-m} \sum_{\mathbf{e}\in I^m} \cos(\mathbf{e}'\boldsymbol{\theta}) \quad \text{and}$$

$$\prod_{j=1}^m \sin\theta_j = 2^{-m}\,(-1)^{\lfloor \frac{m}{2} \rfloor} \begin{cases} \sum_{\mathbf{e}\in I^m} \cos(\mathbf{e}'\boldsymbol{\theta}) p_{\mathbf{e}}, & m \text{ even}, \\ \sum_{\mathbf{e}\in I^m} \sin(\mathbf{e}'\boldsymbol{\theta}) p_{\mathbf{e}}, & m \text{ odd}, \end{cases}$$

and the linear combination of sinusoidal functions

$$a\cos\theta + b\sin\theta = \sqrt{a^2 + b^2}\cos(\theta + \tan^{-1}(-b/a)), \quad a > 0,$$

we have that $\Phi_{m,\delta_\ell}(\boldsymbol{\theta}) = \prod_{j=1}^m g_{\ell,1}(\mathbf{X}_j) + \prod_{j=1}^m g_{\ell,2}(\mathbf{X}_j)$ results in (34). □

B.2 Lemmas of Section 4

The null asymptotic results rely on the asymptotic standard normality of the normalized average of a vector of spherical harmonics of degree lower or equal to K evaluated on the sample given by Lemma B.2. Lemma B.3 gives the asymptotic distribution of a finite general V-statistic, which immediately yields the limit of m-points V-statistics. Lemmas B.4–B.5 are used for the convergence of infinite U-statistics. Finally, Lemma B.7 is a technical result needed for the convergence of infinite V-statistics, and needs Lemma B.6, which gives the expectation of powers of spherical harmonics.

Lemma B.2. *Let $q \geq 1$, $K \geq 1$, $\mathbf{g}_k : \mathbb{S}^q \to \mathbb{R}^{d_{q,k}}$ for $k \leq K$ and $\mathbf{g} : \mathbb{S}^q \to \mathbb{R}^D$ be defined as $\mathbf{g}_k := (g_{k,1}, \ldots, g_{k,d_{q,k}})'$, and $\mathbf{g} := (\mathbf{g}_1', \ldots, \mathbf{g}_K')'$, with $D := \sum_{k=1}^K d_{q,k}$. Then, under $\mathcal{H}_0$ and as $n \to \infty$,*

$$n^{-1/2} \sum_{i=1}^n \mathbf{g}(\mathbf{X}_i) \rightsquigarrow \mathcal{N}_D(0, \mathbf{I}_D). \tag{35}$$

Proof of Lemma B.2. Since $\{\mathbf{X}_i\}_{i=1}^n$ are iid, $\{\mathbf{g}(\mathbf{X}_i)\}_{i=1}^n$ is a sequence of iid vectors. Under $\mathcal{H}_0$, due to the orthonormality of spherical harmonics, we have $\mathrm{E}[\mathbf{g}(\mathbf{X}_1)] = \mathbf{0}$, and $\mathrm{E}[\mathbf{g}(\mathbf{X}_1)\mathbf{g}(\mathbf{X}_1)'] = \mathbf{I}_D$ since $\mathrm{Cov}[g_{k,r}(\mathbf{X}_1), g_{\ell,s}(\mathbf{X}_1)] = \delta_{k\ell}\delta_{rs}$ for all $k, \ell \geq 1$, $1 \leq r \leq d_{q,k}$, $1 \leq s \leq d_{q,\ell}$. Thus, by the multivariate central limit theorem, (35) follows. $\square$

Lemma B.3. *Let $q \geq 1$, $K \geq 1$, $m \geq 2$, w be a real sequence, and $V_{n,m,K}^{(n)}$ be a K-finite general V-statistic given by*

$$V_{n,m,K}^{(n)} := n^{-m/2} \sum_{i_1,\ldots,i_m=1}^n \sum_{k_1,\ldots,k_m=1}^K \sum_{\mathbf{r}} w_{\mathbf{k},\mathbf{r}}\, g_{k_1,r_1}(\mathbf{X}_{i_1}) \cdots g_{k_m,r_m}(\mathbf{X}_{i_m}).$$

Let $n_{\mathbf{k},\mathbf{r}}(k', r') := \sum_{i=1}^m \mathbb{1}_{\{k_i = k', r_i = r'\}}$. Then, under $\mathcal{H}_0$ and as $n \to \infty$,

$$V_{m,w,K}^{(n)} \rightsquigarrow V_{m,w,K}^{\infty} := \sum_{k_1,\ldots,k_m=1}^K \sum_{\mathbf{r}} w_{\mathbf{k},\mathbf{r}} \prod_{k'=1}^K \prod_{r'=1}^{d_{q,k'}} Z_{k',r'}^{n_{\mathbf{k},\mathbf{r}}(k',r')},$$

with $\{Z_{k,r} : k \geq 1, 1 \leq r \leq d_{q,k}\}$ being a collection of independent standard normal random variables.

Proof of Lemma B.3. Note that $V_{m,w,K}^{(n)}$ can be rewritten as

$$\begin{aligned} V_{m,w,K}^{(n)} &= \sum_{k_1,\ldots,k_m=1}^K \sum_{\mathbf{r}} w_{\mathbf{k},\mathbf{r}} \left\{ n^{-m/2} \sum_{i_1,\ldots,i_m=1}^n g_{k_1,r_1}(\mathbf{X}_{i_1}) \cdots g_{k_m,r_m}(\mathbf{X}_{i_m}) \right\} \\ &= \sum_{k_1,\ldots,k_m=1}^K \sum_{\mathbf{r}} w_{\mathbf{k},\mathbf{r}} \left(n^{-1/2} \sum_{i=1}^n g_{k_1,r_1}(\mathbf{X}_i) \right) \cdots \left(n^{-1/2} \sum_{i=1}^n g_{k_m,r_m}(\mathbf{X}_i) \right). \end{aligned}$$

Consider the continuous function $f : \mathbb{R}^D \to \mathbb{R}$, with $D := \sum_{k=1}^K d_{q,k}$, given by $f(\boldsymbol{x}) := \sum_{k_1,\ldots,k_m=1}^K \sum_{\mathbf{r}} w_{\mathbf{k},\mathbf{r}} \prod_{j=1}^m \boldsymbol{x}_{p_j}$ with $p_j := \sum_{k=1}^{k_j-1} d_{q,k} + r_j$. Then, we have $V_{m,w,K}^{(n)} = f\left(n^{-1/2} \sum_{i=1}^n \mathbf{g}(\mathbf{X}_i)\right)$ with $\mathbf{g} : \mathbb{S}^q \to \mathbb{R}^D$ as defined in Lemma B.2, and applying this lemma and the continuous mapping theorem, the result is proven. $\square$

Lemma B.4. *Let $q \geq 1$, $k \geq 1$, $1 \leq r \leq d_{q,k}$, $m \geq 2$ and*

$$W_{m,k,r} := n^{m/2} \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \psi_{k,r}(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_m}).$$

Then, under $\mathcal{H}_0$,

$$\mathrm{E}\big[W_{m,k,r}^{(n)}\, W_{m,\ell,s}^{(n)}\big] = n^m \binom{n}{m}^{-1} \delta_{k\ell}\, \delta_{rs}.$$

Proof of Lemma B.4. Theorem 2 in Section 1.4 of Lee (1990) gives the covariance of two U-statistics. Let $\sigma_{c,c}^2 := \mathrm{Cov}\left[\psi_{c,k,r}(\mathbf{X}_1, \ldots, \mathbf{X}_c), \psi_{c,\ell,s}(\mathbf{X}_1, \ldots, \mathbf{X}_c)\right]$ where $\psi_{c,k,r}\left(\mathbf{x}_1, \ldots, \mathbf{x}_c\right) := \mathrm{E}\left[\psi_{k,r}\left(\mathbf{x}_1, \ldots, \mathbf{x}_c, \mathbf{X}_{c+1}, \ldots, \mathbf{X}_m\right)\right]$ for $c \leq m$. Thus,

$$\mathrm{Cov}\big[W_{m,k,r}^{(n)}, W_{m,\ell,s}^{(n)}\big] = n^m \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \sigma_{c,c}^2. \tag{36}$$

For any $c < m$,

$$\psi_{c,k,r}\left(\mathbf{x}_1, \ldots, \mathbf{x}_c\right) = \prod_{i=1}^{c} g_{k,r}(\mathbf{x}_i) \prod_{j=c+1}^{m} \mathrm{E}\left[g_{k,r}(\mathbf{X}_j)\right],$$

and for $c = m$,

$$\psi_{m,k,r}\left(\mathbf{x}_1, \ldots, \mathbf{x}_m\right) = \psi_{k,r}\left(\mathbf{x}_1, \ldots, \mathbf{x}_m\right).$$

Under $\mathcal{H}_0$, by orthonormality, $\sigma_{c,c}^2 = \delta_{cm}\, \delta_{k\ell}\, \delta_{rs}$, and together with (36), the result follows noting $\mathrm{E}\big[W_{m,k,r}^{(n)}\big] = 0$. □

Lemma B.5. *Let $m \geq 2$, H_m be the mth order Hermite polynomial, and Z_1, Z_2 be two independent standard normal random variables. Let $X_i = Z_i + \mu_i$ with $\mu_i \in \mathbb{R}$ for $i = 1, 2$. Then,*

$$\mathrm{E}\left[H_m(X_1) H_m(X_2)\right] = (\mu_1 \mu_2)^m \quad \text{and} \quad \mathrm{E}\left[H_m(X_1)^2\right] = \sum_{c=0}^{m} \binom{m}{c}^2 c!\, \mu_1^{2(m-c)}.$$

Proof of Lemma B.5. Using the addition formula $H_m(x+y) = \sum_{c=0}^{m} \binom{m}{c} H_c(x) y^{m-c}$,

$$\begin{aligned}
\mathrm{E}\left[H_m(X_1) H_m(X_2)\right] &= \mathrm{E}\left[H_m(Z_1+\mu_1) H_m(Z_2+\mu_2)\right] \\
&= \int_{\mathbb{R}^2} \frac{1}{2\pi} H_m(z+\mu_1) H_m(t+\mu_2)\, e^{-z^2/2}\, e^{-t^2/2}\, \mathrm{d}z\, \mathrm{d}t \\
&= \int_{\mathbb{R}} H_m(z+\mu_1)\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, \mathrm{d}z \int_{\mathbb{R}} H_m(t+\mu_2)\, \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, \mathrm{d}t = (\mu_1 \mu_2)^m,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left[H_m(X_1)^2\right] &= \mathrm{E}\left[H_m(Z_1+\mu_1)^2\right] \\
&= \int_{\mathbb{R}} H_m(z+\mu_1) H_m(z+\mu_1) \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, \mathrm{d}z \\
&= \sum_{c=0}^{m} \sum_{\ell=0}^{m} \binom{m}{c} \binom{m}{\ell} \mu_1^{m-c} \mu_1^{m-\ell} \int_{\mathbb{R}} H_c(z) H_\ell(z)\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, \mathrm{d}z \\
&= \sum_{c=0}^{m} \binom{m}{c}^2 c!\, \mu_1^{2(m-c)}.
\end{aligned}$$

□

Lemma B.6. *Let $q \geq 1$, $k \geq 1$, $1 \leq r \leq d_{q,k}$, $r_{q,k} := \binom{k+q-2}{q-2}$, and ℓ be a nonnegative integer. Let $e_{k,r,\ell} := \mathrm{E}\big[g_{k,r}^{\ell}(\mathbf{X})\big]$. If $\mathbf{X} \sim \nu_q$, then, the following statements hold:*

(i) If $q = 1$, $e_{k,r,\ell} = \begin{cases} 0, & \text{if } \ell \text{ is odd}, \\ 2^{\ell/2}\dfrac{(\ell-1)!!}{\ell!!}, & \text{if } \ell \text{ is even}. \end{cases}$

(ii) If $q \geq 2$, then:

- *(a) $e_{k,r,1} = 0$ and $e_{k,r,2} = 1$, for $k \geq 1$;*
- *(b) $e_{k,r,2\ell+1} = 0$ for k odd;*
- *(c) $e_{k,r,2\ell+1} = 0$ for k even and $((r > r_{q,k}) \vee (\mathbf{m}_r$ is such that $\exists j \leq q-1 : m_j$ is odd$))$, with the corresponding $\mathbf{m}_r$ given in Section 2.*

Proof of Lemma B.6. To prove part (i), recall that for $q = 1$, $d_{1,k} = 2$, $g_{k,1}(\mathbf{x}) = \sqrt{2}\cos(k\theta)$, and $g_{k,2}(\mathbf{x}) = \sqrt{2}\sin(k\theta)$. Using Expression 3.621.3 of Zwillinger et al. (2014) in

$$e_{k,1,\ell} = \frac{2^{\ell/2-1}}{\pi}\int_0^{2\pi}\cos^{\ell}(u)\,\mathrm{d}u, \quad \text{and } e_{k,2,\ell} = \frac{2^{\ell/2-1}}{\pi}\int_0^{2\pi}\sin^{\ell}(u)\,\mathrm{d}u,$$

proves the result.

For part (ii), the orthonormality of the spherical harmonics yields (a). For part (b), since $g_{k,r}$ is a homogeneous polynomial of degree k, it follows that for odd k, $g_{k,r}(-\mathbf{x}) = -g_{k,r}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}^q$. Let $\mathbb{S}^q_- := \{\mathbf{x} \in \mathbb{S}^q : x_1 < 0\}$ and $\mathbb{S}^q_+ := \{\mathbf{x} \in \mathbb{S}^q : x_1 \geq 0\}$, then,

$$\begin{aligned} e_{k,r,2\ell+1} &= \frac{1}{\omega_q}\left(\int_{\mathbb{S}^q_-}(-g_{k,r}(-\mathbf{x}))^{2\ell+1}\,\mathrm{d}\sigma_q(\mathbf{x}) + \int_{\mathbb{S}^q_+} g_{k,r}^{2\ell+1}(\mathbf{x})\,\mathrm{d}\sigma_q(\mathbf{x})\right) \\ &= \frac{1}{\omega_q}\left(-\int_{\mathbb{S}^q_+} g_{k,r}^{2\ell+1}(\mathbf{x})\,\mathrm{d}\sigma_q(\mathbf{x}) + \int_{\mathbb{S}^q_+} g_{k,r}^{2\ell+1}(\mathbf{x})\,\mathrm{d}\sigma_q(\mathbf{x})\right) = 0. \end{aligned}$$

For part (c), integrating in hyperspherical coordinates defined in (3) and using the definition of $g_{k,r}(\mathbf{x})$ gives

$$\begin{aligned} e_{k,r,2\ell+1} &= \int_{\mathbb{S}^q} g_{k,r}^{2\ell+1}(\mathbf{x})\,\nu_q(\mathrm{d}\mathbf{x}) \\ &= \frac{1}{\omega_q}\int_0^{2\pi}\int_0^{\pi}\overset{(q-1)}{\cdots}\int_0^{\pi}\left(\sqrt{B_{\mathbf{m}_r}}\,\zeta_{\mathbf{m}_r}(\theta_1)\prod_{j=1}^{q-1}(\sin\theta_{q-j+1})^{|\mathbf{m}_r^{j+1}|}\,C_{m_j}^{\lambda_j}(\cos\theta_{q-j+1})\right)^{2\ell+1} \\ &\quad\times\prod_{k=2}^{q}\sin^{k-1}\theta_k\,\mathrm{d}\theta_q\cdots\mathrm{d}\theta_2\,\mathrm{d}\theta_1 \\ &= \frac{B_{\mathbf{m}_r}^{\ell+1/2}}{\omega_q}\int_0^{\pi}\overset{(q-1)}{\cdots}\int_0^{\pi}\left[\int_0^{2\pi}\zeta_{\mathbf{m}_r}^{2\ell+1}(\theta_1)\,\mathrm{d}\theta_1\right] \\ &\quad\times\prod_{j=1}^{q-1}(\sin\theta_{q-j+1})^{(2\ell+1)|\mathbf{m}_r^{j+1}|}\left(C_{m_j}^{\lambda_j}(\cos\theta_{q-j+1})\right)^{2\ell+1}\prod_{k=2}^{q}\sin^{k-1}\theta_k\,\mathrm{d}\theta_q\cdots\mathrm{d}\theta_2. \end{aligned}$$

Note that $\zeta_{\mathbf{m}_r}(s)$ depends specifically on $m_{q+1} \in \{0,1\}$. If $m_{q+1} = 0$ and $m_q > 0$, Eq. 2.513.4 in Zwillinger et al. (2014) gives

$$\int_0^{2\pi}\zeta_{\mathbf{m}_r}^{2\ell+1}(\theta_1)\,\mathrm{d}\theta_1 = \int_0^{2\pi}\cos^{2\ell+1}(m_q\theta_1)\,\mathrm{d}\theta_1 = 0, \tag{37}$$

and if $m_{q+1} = 1$, Eq. 2.513.2 yields

$$\int_0^{2\pi} \zeta_{\mathbf{m}_r}^{2\ell+1}(\theta_1)\, \mathrm{d}\theta_1 = \int_0^{2\pi} \sin^{2\ell+1}((m_q+1)\,\theta_1)\, \mathrm{d}\theta_1 = 0.$$

Now, note that (37) is not considered for an $\mathbf{m}$ such that $m_q = m_{q+1} = 0$, which because of the colex order used in the enumeration of $\mathbf{m}_r$, correspond to those such that $r \leq r_{q,k}$, where $r_{q,k}$ is the number of nonnegative tuples of size $q-1$ that sum up to k, as defined in the statement of Lemma B.6. In addition, for those $\mathbf{m}_r$ such that $r \leq r_{q,k}$, we can inspect the symmetries of the associated $g_{k,r}$. Consider the hyperspherical coordinates in (3), such that,

$$\cos\theta_k = \frac{x_{k+1}}{\sqrt{x_1^2 + \cdots + x_{k+1}^2}}, \quad \text{and} \quad \sin\theta_k = \sqrt{\frac{x_1^2 + \cdots + x_k^2}{x_1^2 + \cdots + x_{k+1}^2}},$$

and from the definition of $g_{k,r}(\mathbf{x})$,

$$g_{k,r}(\mathbf{x}) = \sqrt{B_{\mathbf{m}_r}}\, \zeta_{\mathbf{m}_r}(x_1, x_2) \prod_{j=1}^{q-1} \left(\sqrt{\frac{x_1^2 + \cdots + x_{q-j+1}^2}{x_1^2 + \cdots + x_{q-j+2}^2}} \right)^{|\mathbf{m}_r^{j+1}|} C_{m_j}^{\lambda_j} \left(\frac{x_{q-j+2}}{\sqrt{x_1^2 + \cdots + x_{q-j+2}^2}} \right).$$

Note that Gegenbauer polynomials, C_n^α, fulfill $C_n^\alpha(-x) = (-1)^n C_n^\alpha(x)$. Therefore, for any r whose associated $\mathbf{m}_r$ is such that there is a $1 \leq j \leq q-1$ where m_j is odd, $g_{k,r}(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{q+1}) = -g_{k,r}(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{q+1})$, and $e_{k,r,2\ell+1} = 0$ following a similar argument as in the proof of part (b). □

Lemma B.7. *Let $q \geq 1$, $k \geq 1$, $1 \leq r \leq d_{q,k}$, $m \geq 2$. Let $\mathcal{A} := \big\{(\alpha_1, \ldots, \alpha_n) : \big(0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq m\big) \wedge \big(\alpha_i \neq 1, 1 \leq i \leq n\big) \wedge \big(\exists i : \alpha_i \notin \{0,2\}, 1 \leq i \leq n\big) \wedge \big(\sum_{i=1}^n \alpha_i = m\big)\big\}$, and for each $\boldsymbol{\alpha} \in \mathcal{A}$, let $t_{\boldsymbol{\alpha},j} := \sum_{i=1}^n \mathbb{1}_{\{\alpha_i = j\}}$, for $1 \leq j \leq m$, $c_{\boldsymbol{\alpha}} := \sum_{j=1}^m t_{\boldsymbol{\alpha},j}$, and $C_{\boldsymbol{\alpha}} := \dfrac{m!}{\prod_{j=1}^m t_{\boldsymbol{\alpha},j}! \prod_{i=1}^n \alpha_i!}$. Let $e_{k,r,\ell} := \mathrm{E}\big[g_{k,r}^\ell(\mathbf{X}_1)\big]$. Then, under $\mathcal{H}_0$ and as $n \to \infty$:*

(i) If m is even,

$$\mathrm{E}\left[\left(n^{-1/2} \sum_{i=1}^n g_{k,r}(\mathbf{X}_i)\right)^m\right] \sim \frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} R_{m,k,r}, \tag{38}$$

where $R_{m,k,r} := \sum_{\boldsymbol{\alpha} \in \mathcal{A}} n^{c_{\boldsymbol{\alpha}} - m/2 + 1} C_{\boldsymbol{\alpha}} \prod_{i=1}^n e_{k,r,\alpha_i} = O(1)$.

(ii) If m is odd:

(a) For k even, $r \leq r_{q,k}$ with $r_{q,k} := \binom{k+q-2}{q-2}$, and $\mathbf{m}_r$ is such that $\forall j \leq q-1 : m_j$ is even

$$\mathrm{E}\left[\left(n^{-1/2} \sum_{i=1}^n g_{k,r}(\mathbf{X}_i)\right)^m\right] \sim n^{-1/2} \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A} \\ c_{\boldsymbol{\alpha}} = \lfloor m/2 \rfloor}} C_{\boldsymbol{\alpha}} \prod_{i=1}^n e_{k,r,\alpha_i} + n^{-3/2} S_{m,k,r}, \tag{39}$$

where $S_{m,k,r} := \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{A} \\ c_{\boldsymbol{\alpha}} < \lfloor m/2 \rfloor}} n^{c_{\boldsymbol{\alpha}} - (m-3)/2} C_{\boldsymbol{\alpha}} \prod_{i=1}^n e_{k,r,\alpha_i} = O(1)$.

(b) Otherwise, i.e., if k is odd, or $r > r_{q,k}$, or $\mathbf{m}_r$ is such that $\exists j \leq q-1 : m_j$ is odd,

$$\mathrm{E}\left[\left(n^{-1/2} \sum_{i=1}^n g_{k,r}(\mathbf{X}_i)\right)^m\right] = 0. \tag{40}$$

In addition, under $\mathcal{H}_0$, for m even, there exists an $N(\geq 1)$ independent of k and r such that, for all $n \geq N$ and all k, r,

$$\mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right] \leq \frac{m!}{(m/2)!\,2^{m/2-1}} + 2\,n^{-1}R_{m,k,r}.$$

Remark B.1. *When $q = 1$, by Lemma B.6, for m even, e_{k,r,α_i} is indeed independent of r and k, so it is $R_{m,k,r} =: R_m$. In addition, for odd values of m, (39) also simplifies to zero.*

Proof of Lemma B.7. By the multinomial theorem, and the independence of the sample,

$$\begin{aligned}\mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right] &= n^{-m/2}\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\binom{m}{\alpha_1,\alpha_2,\ldots,\alpha_n}\mathrm{E}\left[g_{k,r}^{\alpha_1}(\mathbf{X}_1)\cdots g_{k,r}^{\alpha_n}(\mathbf{X}_n)\right]\\ &= n^{-m/2}\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\binom{m}{\alpha_1,\alpha_2,\ldots,\alpha_n}\prod_{i=1}^{n}\mathrm{E}\left[g_{k,r}^{\alpha_i}(\mathbf{X}_i)\right], \qquad (41)\end{aligned}$$

where the sum is taken over all tuples $\boldsymbol{\alpha} := (\alpha_1,\ldots,\alpha_n)$ of nonnegative integers whose sum is m, that is, $\mathcal{S} := \{(\alpha_1,\ldots,\alpha_n) : (\alpha_i \geq 0,\ 1 \leq i \leq n) \wedge (\sum_{i=1}^n \alpha_i = m)\}$. Since for a given tuple $\boldsymbol{\alpha}$, each of its permutations yields the same results for the expectation, we group those elements, and we take the sum over the set $\mathcal{S}^{\leq} = \{\boldsymbol{\alpha}\in\mathcal{S} : 0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq m\}$. By setting $t_{\boldsymbol{\alpha},j} := \sum_{i=1}^n \mathbb{1}_{\{\alpha_i = j\}}$, $j = 1,\ldots,m$, to be the number of exponents α_i equal to j for a given $\boldsymbol{\alpha}$, we have

$$\begin{aligned}(41) &= n^{-m/2}\sum_{\boldsymbol{\alpha}\in\mathcal{S}^{\leq}}\binom{n}{t_{\boldsymbol{\alpha},0},\ldots,t_{\boldsymbol{\alpha},m}}\binom{m}{\alpha_1,\alpha_2,\ldots,\alpha_n}\prod_{i=1}^{n} e_{k,r,\alpha_i}\\ &= n^{-m/2}\sum_{\boldsymbol{\alpha}\in\mathcal{S}_1^{\leq}}\binom{n}{t_{\boldsymbol{\alpha},0},\ldots,t_{\boldsymbol{\alpha},m}}\binom{m}{\alpha_1,\alpha_2,\ldots,\alpha_n}\prod_{i=1}^{n} e_{k,r,\alpha_i}, \qquad (42)\end{aligned}$$

where in the second equality we used Lemma B.6, part (i) for $q = 1$ and $(ii)(a)$ for $q \geq 2$, and $\mathcal{S}_1^{\leq} := \{\boldsymbol{\alpha}\in\mathcal{S}^{\leq} : \alpha_i \neq 1, 1 \leq i \leq n\}$.

From (41) and using Lemma B.6 parts (i) for $q = 1$, and $(ii)(b$–$c)$ for $q \geq 2$, we immediately get zero when one of the following happens: k is odd, $r > r_{q,k}$, or $\mathbf{m}_r$ is such that $\exists j \leq q-1 : m_j$ is odd, which proves (40).

Then, the number of elements that belong to a certain configuration $\boldsymbol{\alpha}$ is given by $N_{n,\boldsymbol{\alpha}} := \binom{n}{t_{\boldsymbol{\alpha},0},\ldots,t_{\boldsymbol{\alpha},m}}\binom{m}{\alpha_1,\alpha_2,\ldots,\alpha_n}$. Note that we can write $t_{\boldsymbol{\alpha},0} = n - c_{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} := \sum_{j=1}^m t_{\boldsymbol{\alpha},j}$, and for $\boldsymbol{\alpha}$ such that $\alpha_i \neq 1$ for all $1 \leq i \leq n$, then $1 \leq c_{\boldsymbol{\alpha}} \leq m/2$ because elements are grouped at least in pairs.

Let $C_{\boldsymbol{\alpha}} := \dfrac{m!}{\prod_{j=1}^m t_{\boldsymbol{\alpha},j}!\,\prod_{i=1}^n \alpha_i!}$. Then, using Stirling's approximation, as $n \to \infty$,

$$\begin{aligned}n^{-m/2}N_{n,\boldsymbol{\alpha}} &\sim C_{\boldsymbol{\alpha}}\sqrt{\frac{n}{n-c_{\boldsymbol{\alpha}}}}\left(\frac{n}{n-c_{\boldsymbol{\alpha}}}\right)^n\frac{(n-c_{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}}{e^{c_{\boldsymbol{\alpha}}}\,n^{m/2}}\\ &\sim C_{\boldsymbol{\alpha}}\frac{(n-c_{\boldsymbol{\alpha}})^{c_{\boldsymbol{\alpha}}}}{n^{m/2}}\\ &\sim C_{\boldsymbol{\alpha}}n^{c_{\boldsymbol{\alpha}}-m/2}, \qquad (43)\end{aligned}$$

where $\sim$ denotes asymptotic equivalence, and substituting in

$$\begin{aligned}(42) &= \sum_{\boldsymbol{\alpha}\in\mathcal{S}_1^{\leq}} n^{-m/2}\,N_{n,\boldsymbol{\alpha}}\prod_{i=1}^{n} e_{k,r,\alpha_i}\\ &\sim \sum_{\boldsymbol{\alpha}\in\mathcal{S}_1^{\leq}} n^{c_{\boldsymbol{\alpha}}-m/2}\,C_{\boldsymbol{\alpha}}\prod_{i=1}^{n} e_{k,r,\alpha_i}. \qquad (44)\end{aligned}$$

The only possible configuration such that $c_{\boldsymbol{\alpha}} = m/2$ is $\boldsymbol{\alpha} = (\mathbf{0}, 2, \ldots, 2)$, which is feasible when m is even, but not when m is odd. Hence, for even m, and denoting $\mathcal{S}^* := \{\boldsymbol{\alpha} \in \mathcal{S}_1^{\leq} : \exists i : \alpha_i \notin \{0, 2\}, 1 \leq i \leq n\}$ we have

$$
\begin{aligned}
(44) &= C_{(2,\ldots,2)} \prod_{j=1}^{n} \mathrm{E}\left[g_{k,r}^2(\mathbf{X}_j)\right] + \sum_{\boldsymbol{\alpha} \in \mathcal{S}^*} n^{c_{\boldsymbol{\alpha}} - m/2} C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i} \\
&= \frac{m!}{(m/2)!\, 2^{m/2}} + \sum_{\boldsymbol{\alpha} \in \mathcal{S}^*} n^{c_{\boldsymbol{\alpha}} - m/2} C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i} \\
&= \frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} \sum_{\boldsymbol{\alpha} \in \mathcal{S}^*} n^{c_{\boldsymbol{\alpha}} - m/2 + 1} C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i},
\end{aligned}
$$

where in the second equality we used Lemma B.6(a). However, for m odd,

$$
(44) = n^{-1/2} \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{S}_1^{\leq} \\ c_{\boldsymbol{\alpha}} = \lfloor m/2 \rfloor}} C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i} + n^{-3/2} \sum_{\substack{\boldsymbol{\alpha} \in \mathcal{S}_1^{\leq} \\ c_{\boldsymbol{\alpha}} < \lfloor m/2 \rfloor}} n^{c_{\boldsymbol{\alpha}} - (m-3)/2} C_{\boldsymbol{\alpha}} \prod_{i=1}^{n} e_{k,r,\alpha_i}.
$$

For every $\boldsymbol{\alpha} \in \mathcal{S}_1^{\leq}$ such that $c_{\boldsymbol{\alpha}} < m/2$, there are less than $m/2$ terms with $\alpha_i > 0$ in $\prod_{i=1}^{n} e_{k,r,\alpha_i}$, and for the rest of terms, $e_{k,r,0} = 1$. Therefore, $\prod_{i=1}^{n} e_{k,r,\alpha_i} = O(1)$.

To prove the second part, note that Stirling's approximation used in (43) ensures that for all $\epsilon > 0$, there exists an $N_{\boldsymbol{\alpha}}$, independent of k and r, such that for all $n \geq N_{\boldsymbol{\alpha}}$,

$$
\left| \frac{n^{-m/2} N_{n,\boldsymbol{\alpha}}}{C_{\boldsymbol{\alpha}} n^{c_{\boldsymbol{\alpha}} - m/2}} - 1 \right| < \epsilon.
$$

Letting $\epsilon = 1$ and setting $N := \max\{N_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{A}\}$ completes the proof. □

Remark B.2. *Lemma B.7 gives an asymptotic expansion of the moments of the normed sum of a spherical harmonic as $n \to \infty$. An exact expression for the moments in the central limit theorem, which applies to* (38) *was given in von Bahr (1965). However, this approach is not applied for two reasons: (i) this expression does not improve the bounds used in Proposition 4.2, because the dominant moment in the $(m-2)$th term of* (45) *is $e_{k,r,m}$, and (ii) the terms in* (45) *are complicated to compute as m increases, since they involve computing explicitly the integer partitions of all even integers $0 < j < m$. Nevertheless, it is interesting to see how our asymptotic approximation and the exact moments are related. According to Theorem 1 in von Bahr (1965), for even values of $m \geq 2$,*

$$
\begin{aligned}
\mathrm{E}\left[\left(n^{-1/2} \sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^m\right] &= \frac{m!}{(m/2)!\, 2^{m/2}} + \sum_{j=1}^{m-2} n^{-j/2} \int_{\mathbb{R}} x^m \mathrm{d}P_j(-\Phi)(x) \\
&= \frac{m!}{(m/2)!\, 2^{m/2}} + n^{-1} \int_{\mathbb{R}} x^m \mathrm{d}P_2(-\Phi)(x) \\
&\quad + n^{-2} \int_{\mathbb{R}} x^m \mathrm{d}P_4(-\Phi)(x) + \cdots \\
&\quad + n^{2-m/2} \int_{\mathbb{R}} x^m \mathrm{d}P_{m-4}(-\Phi)(x) \\
&\quad + n^{1-m/2} \int_{\mathbb{R}} x^m \mathrm{d}P_{m-2}(-\Phi)(x), \qquad (45)
\end{aligned}
$$

where

$$
\mathrm{d}P_j(-\Phi)(x) = \sum \prod_i \frac{1}{k_i!} \left(\frac{\lambda_{l_i}}{\ell_i!}\right)^{k_i} (-1)^s \phi^{(s)}(x)\, \mathrm{d}x,
$$

with the summation being over all the integer partitions of j, the values k_i being defined such that $\sum_i ik_i = j$, $\ell_i := i+2$, $s := \sum_i k_i\ell_i$, λ_ℓ denotes the ℓth cumulant of $g_{k,r}(\mathbf{X})$, and $\phi^{(s)}$ is the sth derivative of the pdf of a standard normal distribution. The sum contains only integer powers of n due to the symmetry of $P_j(-\Phi)$.

For concreteness, let us investigate the case $m = 4$,

$$\mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^4\right] = 3 + n^{-1}(e_{k,r,4} - 3). \tag{46}$$

Following the proof of Lemma B.7, we can compute explicitly the moments, which coincide with (46). Applying Stirling's approximation to $N_{n,\boldsymbol{\alpha}}$, we end up with the approximation of Lemma B.7,

$$\mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^4\right] \sim 3 + n^{-1}\, e_{k,r,4}. \tag{47}$$

Thus, the difference between (46) and (47) is $O(n^{-1})$.

The conclusion is analogous with $m = 6$,

$$\begin{aligned}\mathrm{E}\left[\left(n^{-1/2}\sum_{i=1}^{n} g_{k,r}(\mathbf{X}_i)\right)^6\right] &= 15 + n^{-1}\left(15\,(e_{k,r,4} - 3) + 10e_{k,r,3}^2\right)\\ &\quad + n^{-2}\left(e_{k,r,6} - 15e_{k,r,4} - 10e_{k,r,3}^2 + 30\right)\\ &\sim 15 + n^{-1}\left(15\,e_{k,r,4} + 10\,e_{k,r,3}^2\right) + n^{-2}\,e_{k,r,6}.\end{aligned}$$

B.3 Lemmas of Section 5

Lemma B.9 is used to prove the consistency against fixed alternatives. Analogously to the null asymptotic results, the asymptotic results under local alternatives depend on the normal limit given in Lemma B.10, which, in turn, needs Lemma B.8. For rotationally symmetric alternatives, Lemma B.11 is also required.

Lemma B.8. *Let $q \geq 1$, $k \geq 1$, $\mathbf{g}_k : \mathbb{S}^q \to \mathbb{R}^{d_{q,k}}$ be as defined in Lemma B.2, and $\mathbf{X} \sim h_n$ for $n \geq 1$, with h_n as defined in (14). Assume the series $\sum_{k=1}^{\infty} \mathbf{h}_k'\mathbf{g}_k(\mathbf{x})$ converges uniformly in $\mathbb{S}^q$. Then,*

$$\begin{aligned}\mathrm{E}_{h_n}\left[\mathbf{g}_k(\mathbf{X})\right] &= n^{-1/2}\mathbf{h}_k\\ \mathrm{E}_{h_n}\left[\mathbf{g}_k(\mathbf{X})\mathbf{g}_\ell(\mathbf{X})'\right] &= \delta_{k\ell}\mathbf{I}_{d_{q,k},d_{q,\ell}} + n^{-1/2}\mathbf{A}_{k,\ell}(h),\end{aligned}$$

where $\mathbf{A}_{k,\ell}(h) := \sum_{m=1}^{k+\ell}\int_{\mathbb{S}^q} \mathbf{h}_m'\mathbf{g}_m(\mathbf{x})\mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})'\,\mathrm{d}\nu_q(\mathbf{x})$.

Proof of Lemma B.8. By straightforward computation, we obtain

$$\begin{aligned}\mathrm{E}_{h_n}\left[\mathbf{g}_k(\mathbf{X})\right] &= \int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})h_n(\mathbf{x})\,\mathrm{d}\sigma_q(\mathbf{x})\\ &= \int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\left\{1 + n^{-1/2}\sum_{k=1}^{\infty}\mathbf{h}_k'\mathbf{g}_k(\mathbf{x})\right\}\mathrm{d}\nu_q(\mathbf{x})\\ &= n^{-1/2}\int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\sum_{\ell=1}^{\infty}\mathbf{h}_\ell'\mathbf{g}_\ell(\mathbf{x})\,\mathrm{d}\nu_q(\mathbf{x})\\ &= n^{-1/2}\sum_{\ell=1}^{\infty}\int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})'\,\mathrm{d}\nu_q(\mathbf{x})\mathbf{h}_\ell\end{aligned}$$

$$= n^{-1/2} \int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\mathbf{g}_k(\mathbf{x})' \,\mathrm{d}\nu_q(\mathbf{x})\, \mathbf{h}_k$$
$$= n^{-1/2}\mathbf{h}_k,$$

and

$$\begin{aligned}
\mathrm{E}_{h_n}\left[\mathbf{g}_k(\mathbf{X})\mathbf{g}_\ell(\mathbf{X})'\right] &= \int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})' h_n(\mathbf{x}) \,\mathrm{d}\sigma_q(\mathbf{x}) \\
&= \int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})' \,\mathrm{d}\nu_q(\mathbf{x}) \\
&\quad + n^{-1/2} \int_{\mathbb{S}^q} \mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})' \sum_{m=1}^{\infty} \mathbf{h}'_m \mathbf{g}_m(\mathbf{x}) \,\mathrm{d}\nu_q(\mathbf{x}) \\
&= \delta_{k\ell}\mathbf{I}_{d_{q,k},d_{q,\ell}} + n^{-1/2} \sum_{m=1}^{\infty} \int_{\mathbb{S}^q} \mathbf{h}'_m \mathbf{g}_m(\mathbf{x})\mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})' \,\mathrm{d}\nu_q(\mathbf{x}) \\
&= \delta_{k\ell}\mathbf{I}_{d_{q,k},d_{q,\ell}} + n^{-1/2}\mathbf{A}_{k,\ell}(h),
\end{aligned}$$

where $\mathbf{A}_{k,\ell}(h) := \sum_{m=1}^{k+\ell} \int_{\mathbb{S}^q} \mathbf{h}'_m \mathbf{g}_m(\mathbf{x})\mathbf{g}_k(\mathbf{x})\mathbf{g}_\ell(\mathbf{x})' \,\mathrm{d}\nu_q(\mathbf{x})$ and the sum is truncated since spherical harmonics of degree $m > k+\ell$ are orthogonal to polynomials $\mathbf{x} \mapsto g_{k,r}(\mathbf{x})g_{\ell,s}(\mathbf{x})$ of lower degree. □

Lemma B.9. *Let $q \geq 1$, $k \geq 1$, $1 \leq r \leq d_{q,k}$, and $T_{n,k,r} := n^{-1}\sum_{i=1}^n g_{k,r}(\mathbf{X}_i)$. Then, under H and as $n \to \infty$, with $h_{k,r}$ denoting the rth element of $\mathbf{h}_k$, the following statements hold:*

(i) If $h_{k,r} < 0$, $\sqrt{n}T_{n,k,r} \xrightarrow{\mathrm{P}} -\infty$.

(ii) If $h_{k,r} = 0$, $\sqrt{n}T_{n,k,r} \rightsquigarrow \mathcal{N}(0, \sigma^2_{k,r})$, with $\sigma^2_{k,r} := 1 + (\mathbf{A}_{k,k}(h))_{r,r}$ and $\mathbf{A}_{k,k}(h)$ given in Lemma B.8.

(iii) If $h_{k,r} > 0$, $\sqrt{n}T_{n,k,r} \xrightarrow{\mathrm{P}} +\infty$.

Proof of Lemma B.9. Analogous to Lemma B.8, since the series $\sum_{k=1}^\infty \mathbf{h}'_k \mathbf{g}_k(\mathbf{x})$ is assumed to converge uniformly in $\mathbb{S}^q$, we obtain

$$\mathrm{E}_H\left[\mathbf{g}_k(\mathbf{X}_1)\right] = \mathbf{h}_k$$
$$\mathrm{E}_H\left[\mathbf{g}_k(\mathbf{X}_1)\mathbf{g}_\ell(\mathbf{X}_1)'\right] = \delta_{k,\ell}\mathbf{I}_{d_{q,k}d_{q,\ell}} + \mathbf{A}_{k,\ell}(h).$$

In case (ii), the result is immediate by CLT. In cases (i) and (iii), we can write

$$\sqrt{n}T_{n,k,r} = \sqrt{n}\left(T_{n,k,r} - h_{k,r}\right) + \sqrt{n}h_{k,r} =: A_{n,k,r} + B_{n,k,r}$$

Then, $A_{n,k,r} \rightsquigarrow \mathcal{N}(0, \sigma^2_{k,r})$ and $A_{n,k,r} = O_{\mathrm{P}}(1)$. Note also that $B_{n,k,r} \to \mathrm{sgn}(h_{k,r}) \cdot \infty$. Thus, the result is proven. □

Lemma B.10. *Let $q \geq 1$, $K \geq 1$, $\mathbf{h} = (\mathbf{h}'_1, \ldots, \mathbf{h}'_K)'$, $\mathbf{g}_k : \mathbb{S}^q \to \mathbb{R}^{d_{q,k}}$ for $k \leq K$ and $\mathbf{g} : \mathbb{S}^q \to \mathbb{R}^D$ be as defined in Lemma B.2. Then, under h_n and as $n \to \infty$,*

$$n^{-1/2}\sum_{i=1}^n \mathbf{g}(\mathbf{X}_i) \rightsquigarrow \mathcal{N}_D(\mathbf{h}, \mathbf{I}_D).$$

Proof of Lemma B.10. Let $\mathbf{Z}_i := (\mathbf{g}_1(\mathbf{X}_i)', \ldots, \mathbf{g}_K(\mathbf{X}_i)')'$, $\mathbf{T}_n := n^{-1}\sum_{i=1}^n \mathbf{Z}_i$, and $D = \sum_{k=1}^K d_{q,k}$.

We first prove that $\sqrt{n}\,(\mathbf{T}_n - \mathrm{E}_{h_n}[\mathbf{T}_n]) \rightsquigarrow \mathcal{N}_D(\mathbf{0}, \mathbf{I}_D)$, which by the Cramér–Wold theorem is equivalent to prove that

$$\sqrt{n}\left(\mathbf{c}'\mathbf{T}_n - \mathrm{E}_{h_n}\left[\mathbf{c}'\mathbf{T}_n\right]\right) \rightsquigarrow \mathcal{N}(0,1), \quad \text{for every } \mathbf{c} \in \mathbb{S}^{D-1}. \tag{48}$$

We have that

$$\sqrt{n}\left(\mathbf{c}'\mathbf{T}_n - \mathrm{E}_{h_n}\left[\mathbf{c}'\mathbf{T}_n\right]\right) = n^{-1/2}\left(\sum_{i=1}^{n}\mathbf{c}'\mathbf{Z}_i - \mathrm{E}_{h_n}\left[\sum_{i=1}^{n}\mathbf{c}'\mathbf{Z}_i\right]\right) =: \sum_{i=1}^{n} X_{n,i}$$

and

$$\begin{aligned} s_n^2 := \mathrm{Var}_{h_n}\left[\sum_{i=1}^{n} X_{n,i}\right] &= n^{-1}\mathrm{Var}_{h_n}\left[\sum_{i=1}^{n}\mathbf{c}'\mathbf{Z}_i\right] \\ &= \mathbf{c}'\,\mathrm{Var}_{h_n}[\mathbf{Z}_1]\,\mathbf{c} \\ &= \mathbf{c}'\mathrm{diag}\left(\mathbf{I}_{d_{q,1}}, \ldots, \mathbf{I}_{d_{q,K}}\right)\mathbf{c} + O(n^{-1/2}) \\ &= 1 + O(n^{-1/2}), \end{aligned}$$

due to $\mathbf{Z}_i$ being iid and Lemma B.8. Lyapunov's condition for $\delta = 1$,

$$\lim_{n\to\infty}\frac{1}{s_n^3}\sum_{i=1}^{n}\mathrm{E}_{h_n}\left[|X_{n,i}|^3\right] \leq \frac{8D^{3/2}C^3}{n^{1/2}s_n^3} \to 0,$$

holds since

$$\begin{aligned} |X_{n,i}| &= \frac{1}{\sqrt{n}}\left|\mathbf{c}'\mathbf{Z}_i - \mathrm{E}_{h_n}\left[\mathbf{c}'\mathbf{Z}_i\right]\right| \leq \frac{1}{\sqrt{n}}\left(\left|\mathbf{c}'\mathbf{Z}_i\right| + \left|\mathrm{E}_{h_n}\left[\mathbf{c}'\mathbf{Z}_i\right]\right|\right) \\ &\leq \frac{1}{\sqrt{n}}\left(\|\mathbf{c}\|\,\|\mathbf{Z}_i\| + \mathrm{E}_{h_n}\left[\left|\mathbf{c}'\mathbf{Z}_i\right|\right]\right) \leq \frac{1}{\sqrt{n}}\left(\|\mathbf{c}\|\,\|\mathbf{Z}_i\| + \mathrm{E}_{h_n}[\|\mathbf{c}\|\,\|\mathbf{Z}_i\|]\right) \\ &= \frac{1}{\sqrt{n}}\left(\|\mathbf{Z}_i\| + \mathrm{E}_{h_n}[\|\mathbf{Z}_i\|]\right) \leq 2\frac{\sqrt{D}C}{\sqrt{n}}, \end{aligned}$$

where $C := \max_{1\leq k\leq K}\max_{1\leq r\leq d_{q,k}}\sup_{\mathbf{x}\in\mathbb{S}^q}|g_{k,r}(\mathbf{x})|$ is well-defined because $g_{k,r} : \mathbb{S}^q \to \mathbb{R}$ is a continuous function defined on the compact set $\mathbb{S}^q$, and, thus, it attains its maximum. Therefore, by Lyapunov CLT, (48) is proved.

Second, we obtain the limit distribution of

$$\sqrt{n}\mathbf{T}_n = \sqrt{n}\,(\mathbf{T}_n - \mathrm{E}_{h_n}[\mathbf{T}_n]) + \sqrt{n}\mathrm{E}_{h_n}[\mathbf{T}_n] =: \mathbf{V}_n + \mathbf{W}_n.$$

By (48), $\mathbf{V}_n \rightsquigarrow \mathcal{N}(0,1)$, and by Lemma B.8, $\mathbf{W}_n = (\mathbf{h}_1', \ldots, \mathbf{h}_K')'$, thus $\sqrt{n}\mathbf{T}_n \rightsquigarrow \mathcal{N}_D(\mathbf{h}, \mathbf{I}_D)$. □

Lemma B.11. *Let $q \geq 1$, $k \geq 1$, and $\mathbf{g}_k : \mathbb{S}^q \to \mathbb{R}^{d_{q,k}}$ be as defined in Lemma B.2. Then, under $\mathrm{P}^{(n)}_{\kappa_n,f}$,*

$$\mathrm{E}[\mathbf{g}_k(\mathbf{X})] = \frac{1}{h_{q,k}(1)}\mathrm{E}\left[C_k^{(q-1)/2}(\mathbf{X}'\boldsymbol{\mu})\right]\mathbf{g}_k(\boldsymbol{\mu}).$$

Proof of Lemma B.11. The result follows from applying the Funk-Hecke Theorem,

$$\begin{aligned} \mathrm{E}[g_{k,r}(\mathbf{X})] &= c_{q,\kappa_n,f}\int_{\mathbb{S}^q} g_{k,r}(\mathbf{x})f(\kappa_n\mathbf{x}'\boldsymbol{\mu})\,\mathrm{d}\sigma_q(\mathbf{x}) \\ &= c_{q,\kappa_n,f}\frac{\omega_{q-1}}{h_{q,k}(1)}g_{k,r}(\boldsymbol{\mu})\int_{-1}^{1}C_k^{(q-1)/2}(s)f(\kappa_n s)(1-s^2)^{q/2-1}\,\mathrm{d}s \\ &= \frac{1}{h_{q,k}(1)}\mathrm{E}\left[C_k^{(q-1)/2}(\mathbf{X}'\boldsymbol{\mu})\right]g_{k,r}(\boldsymbol{\mu}). \end{aligned}$$

□

C Additional simulation results

C.1 Power under fixed alternatives

Sections 8.2 and 8.3 show power simulations of the proposed m-points tests and their quasi-rotation-invariant versions under the fixed alternative scenarios $(*)$, $(\circ)$, and $(+)$.

In Section 8.2, the figures show the difference between the empirical power attained by an $m > 2$ test compared to the corresponding Sobolev test ($m = 2$). Figures 5–7 show the empirical rejection proportions (empirical power) in absolute terms of m-points tests based on $V_{m,w,10}$, with the parameters specified in Section 8.2. In particular, the figures correspond to results shown in Figures 1–3, respectively.

In Section 8.3, only summarized results of the experiment are included. Figures 8–10 display the empirical power gains of quasi-rotation-invariant V-tests, $p^{\mathrm{HMP},R}_{m,w,10}$ with $R = 50$, relative to Sobolev tests ($m = 2$) under the same simulation setting as in Section 8.2. Figures 11–13 show the corresponding empirical power.

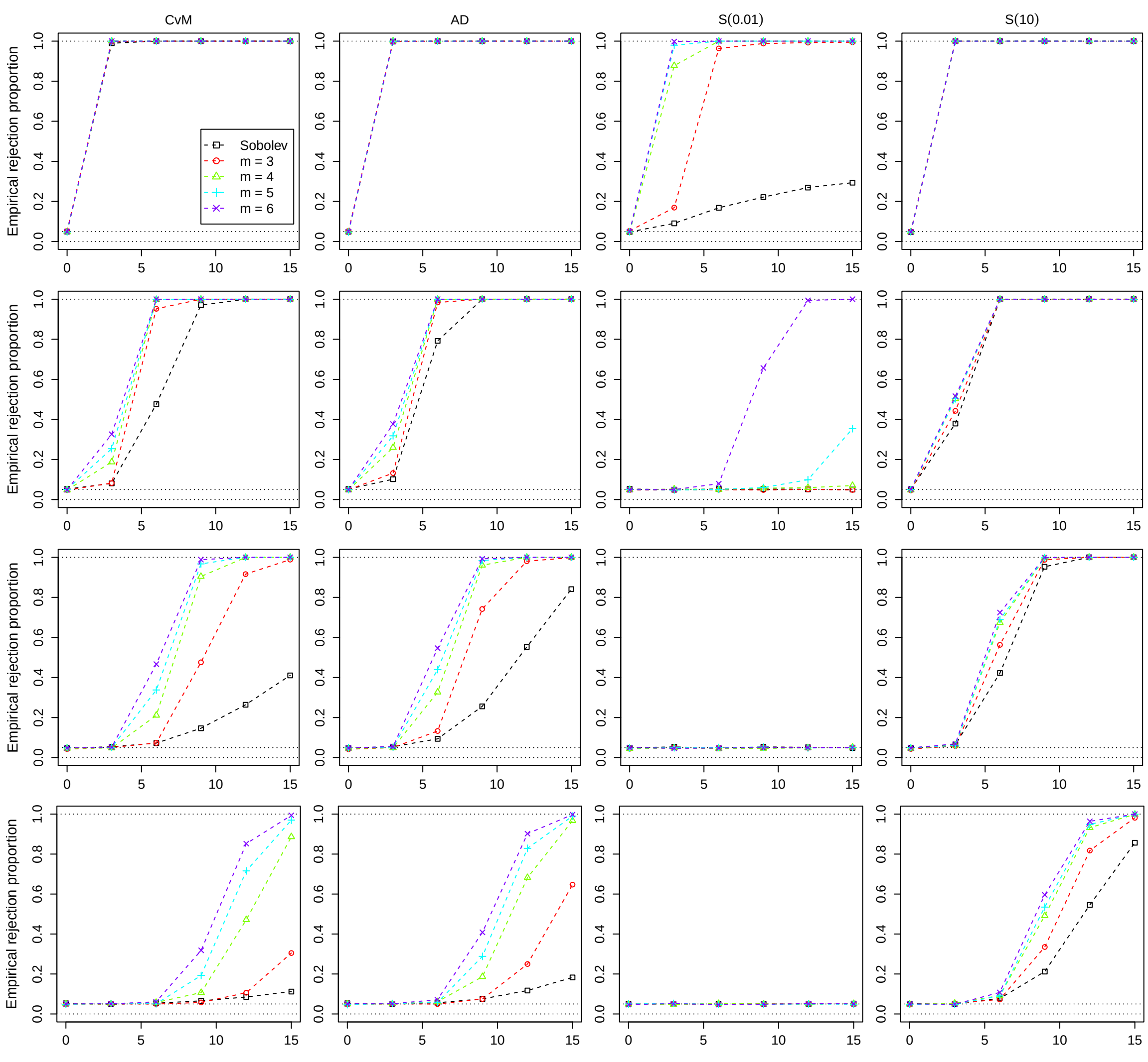


Figure 5: Empirical rejection proportion curves under scenario $(*)$ as a function of concentration κ, each row corresponding to a different number of mixture components, $N \in \{2, 3, 4, 5\}$. Curves display the power of m-points V-tests and the corresponding Sobolev test ($m = 2$). Tests are based on $V_{m,w,10}$, $m \in \{3, 4, 5, 6\}$ with weights indicated by columns, and $n = 100$.

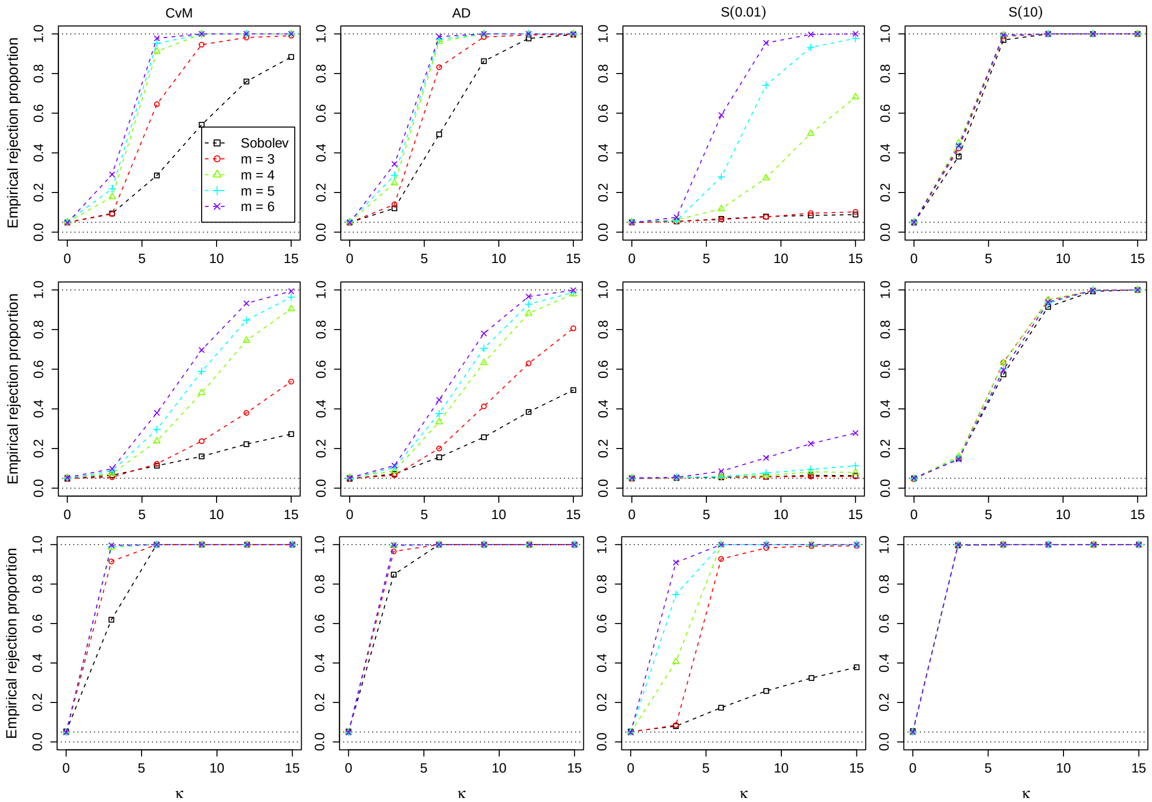


Figure 6: Empirical rejection proportion curves under scenario ($\circ$) as a function of concentration κ, each row corresponding to a different parameter value $\theta \in \{\pi/12, \pi/4, 5\pi/12\}$. The same description of Figure 5 applies.

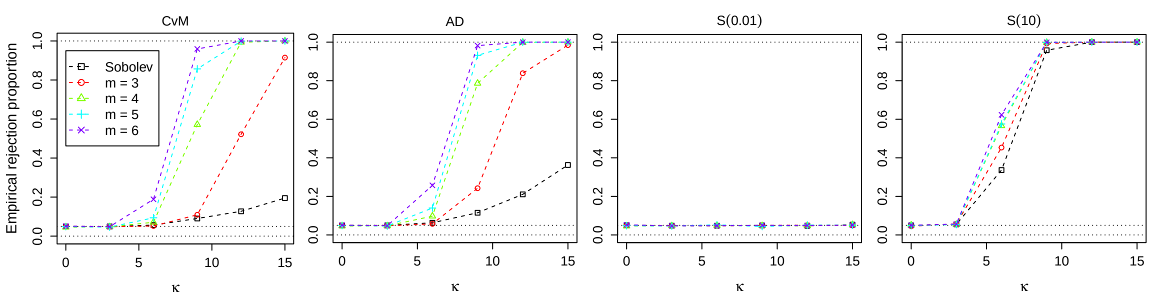


Figure 7: Empirical rejection proportion curves under scenario ($+$) as a function of concentration κ. The same description of Figure 5 applies.

C.2 Asymptotic distribution under rotated local alternatives

Section 5.2 presents the asymptotic distribution of m-points U- and V-statistics under local alternatives. As noted in Remark 5.1, the asymptotic distribution for $m = 2$ is invariant under rotations of the alternative distribution. However, for $m > 2$, this invariance is not guaranteed. In this section, we present several numerical experiments illustrating the asymptotic distribution for each dimension $q \in \{1, 2\}$ under alternatives of the form (15), including multiple rotated versions of each alternative.

We explore two rotationally symmetric local alternatives, with a base pdf given by:

(*i*) (*vMF*) the vMF distribution, with pdf $f_{\boldsymbol{\mu},\kappa}$ defined in ($\vee$).

(*ii*) (*Watson*) the Watson distribution, given by $f_{\mathrm{W},\boldsymbol{\mu},\kappa} : \mathbb{S}^q \to \mathbb{R}$, such that $\mathbf{x} \mapsto c_{q,\kappa} e^{\kappa(\mathbf{x}'\boldsymbol{\mu})^2}$, where $c_{q,\kappa}$ is the corresponding normalizing constant.

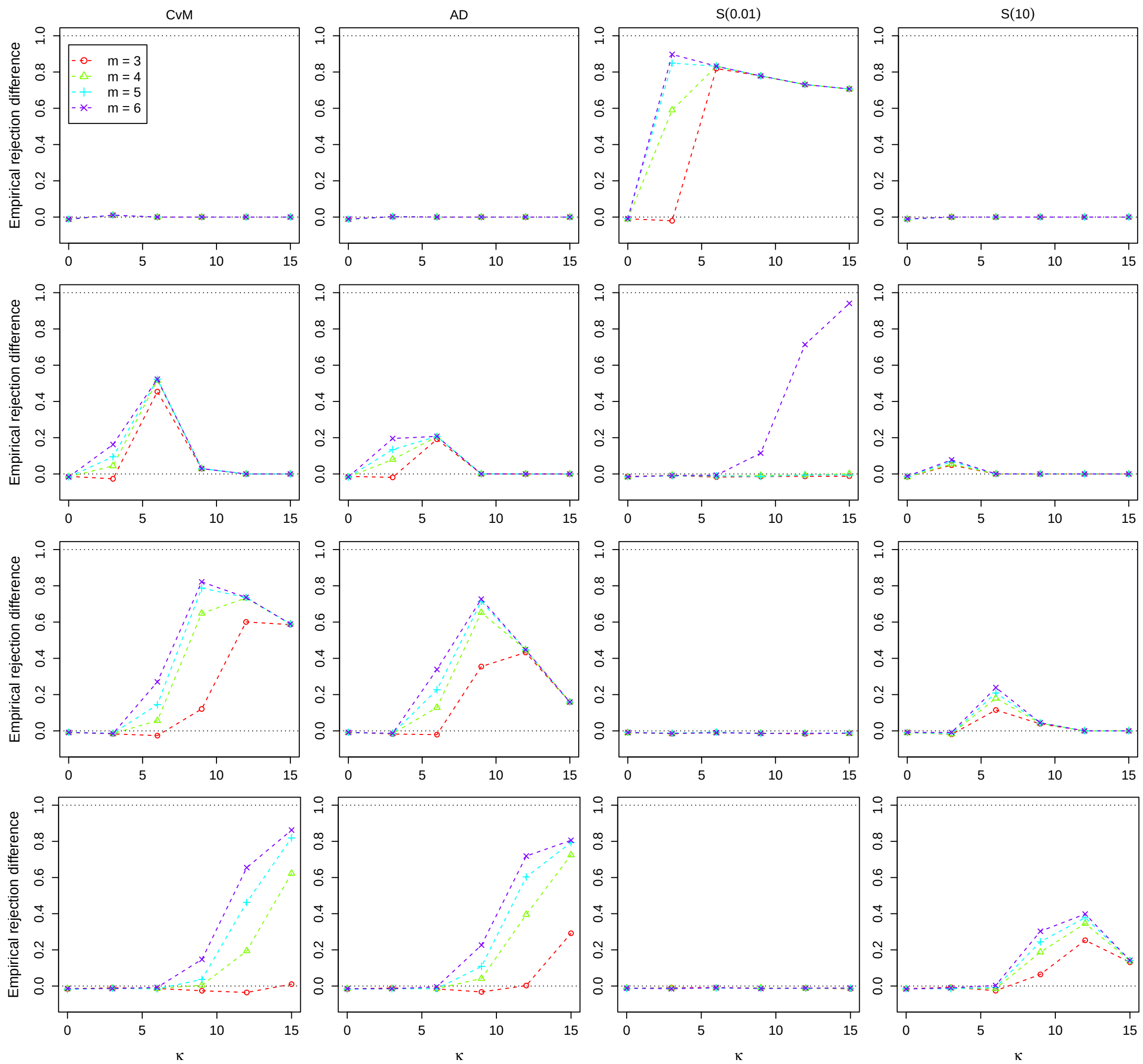


Figure 8: Empirical rejection proportion difference curves under scenario $(*)$ as a function of concentration κ, each row corresponding to a different number of mixture components, $N \in \{2, 3, 4, 5\}$. Curves compare m-points V-tests against the baseline Sobolev test ($m = 2$). Tests are the quasi-rotation-invariant $p^{\mathrm{HMP},50}_{m,w,10}$ V-tests, $m \in \{3, 4, 5, 6\}$ with weights indicated by columns, and $n = 100$.

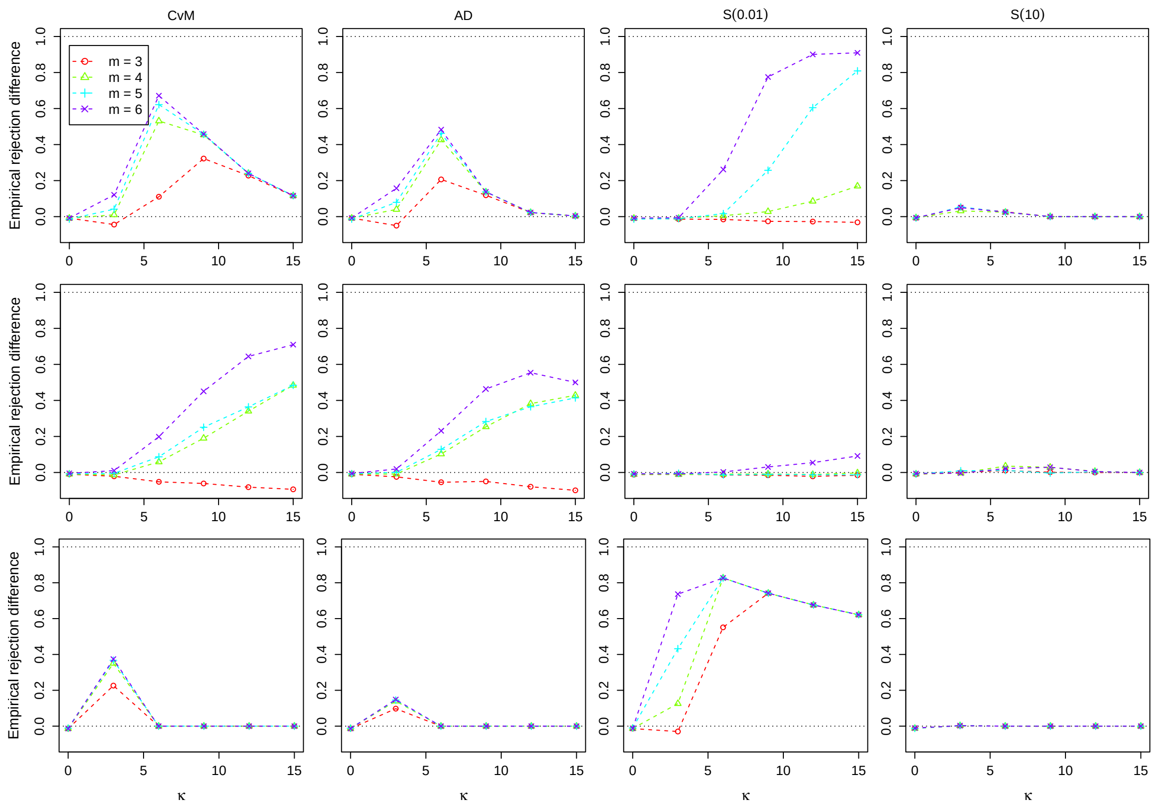


Figure 9: Empirical rejection proportion difference curves under scenario ($\circ$) as a function of concentration κ, each row corresponding to a different parameter value $\theta \in \{\pi/12, \pi/4, 5\pi/12\}$. The same description of Figure 8 applies.

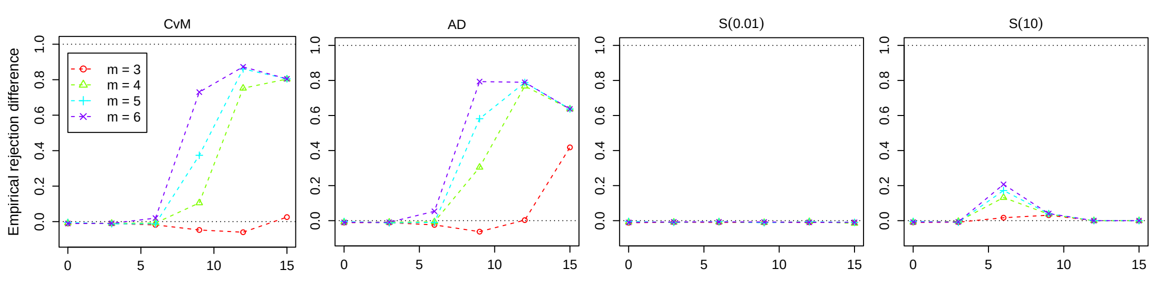


Figure 10: Empirical rejection proportion difference curves under scenario (+) as a function of concentration κ. The same description of Figure 8 applies.

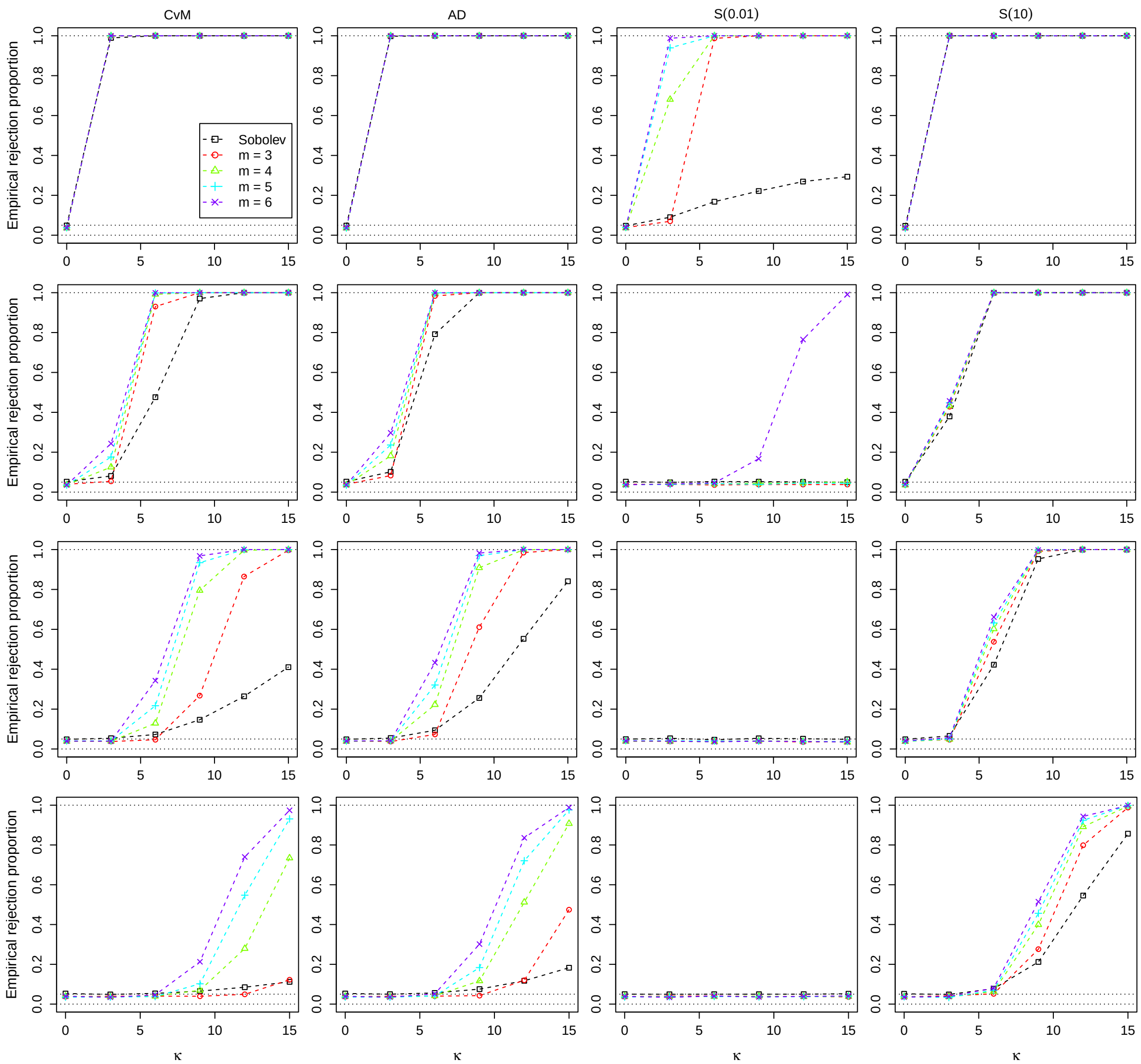


Figure 11: Empirical rejection proportion curves under scenario $(*)$ as a function of concentration κ, each row corresponding to a different number of mixture components, $N \in \{2, 3, 4, 5\}$. Curves display the power of m-points V-tests and the corresponding Sobolev test ($m = 2$). Tests are the quasi-rotation-invariant $p_{m,w,10}^{\mathrm{HMP},50}$ V-tests, $m \in \{3, 4, 5, 6\}$ with weights indicated by columns, and $n = 100$.

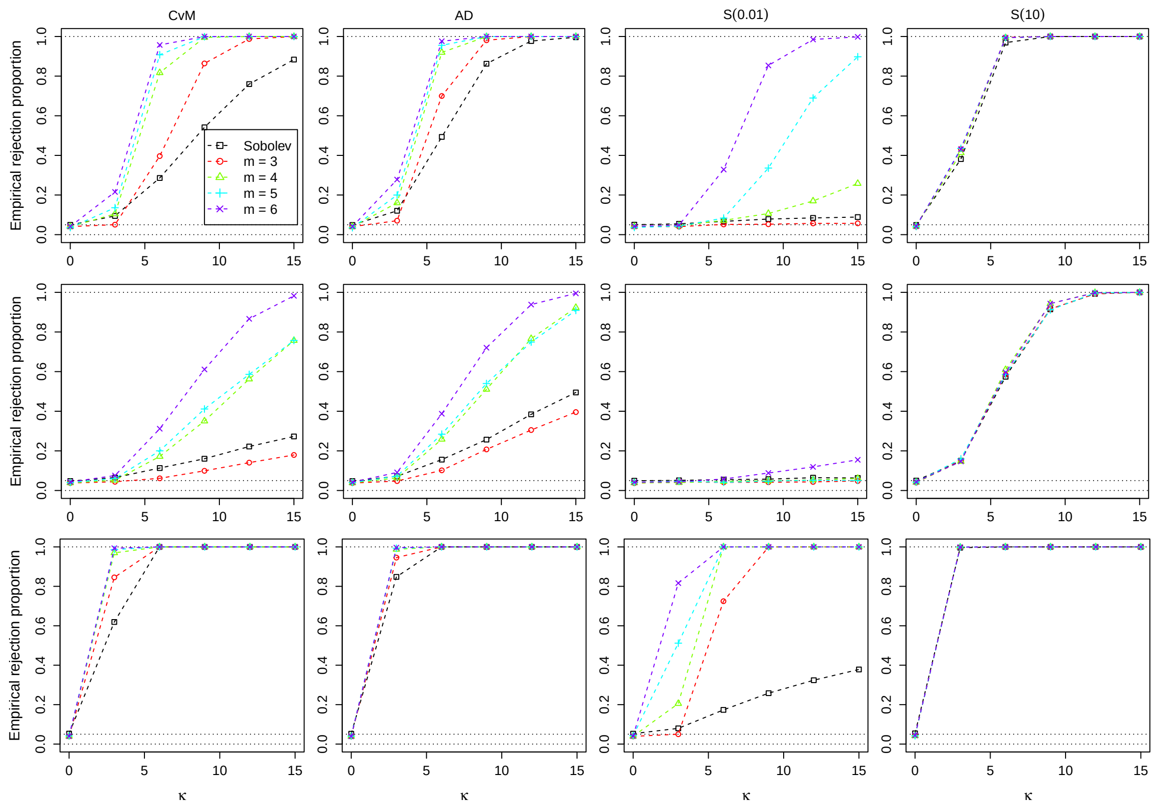


Figure 12: Empirical rejection proportion curves under scenario (∘) as a function of concentration κ, each row corresponding to a different parameter value $\theta \in \{\pi/12, \pi/4, 5\pi/12\}$. The same description of Figure 11 applies.

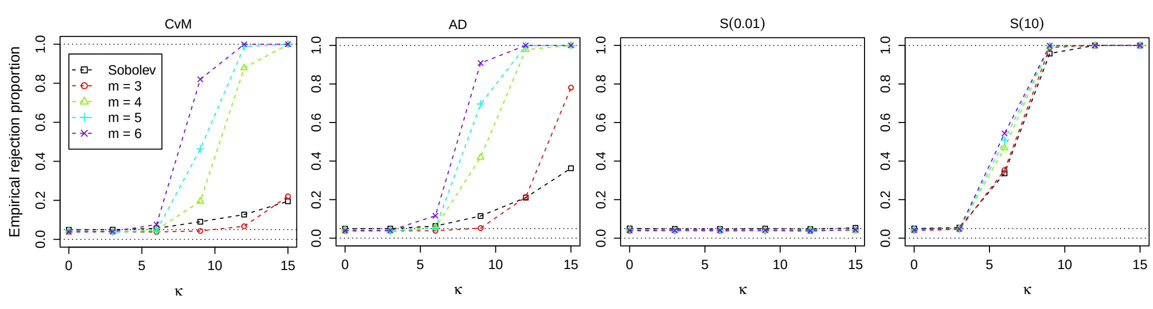


Figure 13: Empirical rejection proportion curves under scenario (+) as a function of concentration κ. The same description of Figure 11 applies.

Note that, due to rotational symmetry, rotating any of these densities with a location parameter $\boldsymbol{\mu}$ results in the same functional form with a different location parameter $\boldsymbol{\mu}'$.

For each alternative scenario described in (i) and (ii), and for dimensions $q \in \{1, 2\}$, we consider concentration $\kappa = 6$ and the following set of location parameters:

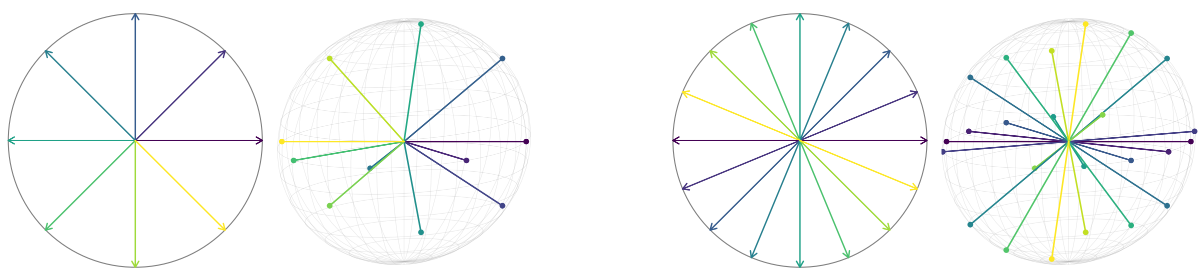
(a) For $q = 1$, $\boldsymbol{\mu} = (\cos\theta, \sin\theta)'$ with

- (i) $\theta \in \{k\pi/4 : k = 0, 1, \ldots, 7\}$ for vMF, and
- (ii) $\theta \in \{k\pi/8 : k = 0, 1, \ldots, 7\}$ for Watson.

(b) For $q = 2$, $\boldsymbol{\mu} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)'$ with

- (i) $(\theta, \phi) \in \{(0, 0), (\pi, 0)\} \cup (\{\pi/4, \pi/2, 3\pi/4\} \times \{-\pi, -\pi/3, \pi/3\})$ for vMF, and
- (ii) $(\theta, \phi) \in \{(0, 0), (\pi/8, 0), (\pi/8, -\pi)\} \cup (\{\pi/4, 3\pi/8, \pi/2\} \times \{-\pi, -\pi/3, \pi/3\})$ for Watson.

All considered locations are illustrated in Figure 14. For each statistic, $U_{m,w,10}$ and $V_{m,w,10}$, with $m \in \{2, 3, 4, 5, 6\}$ and weights presented in Table 1, we display the histograms of the asymptotic distributions corresponding to each location parameter $\boldsymbol{\mu}$, with each distribution shown in a different color. For reference, the null asymptotic distribution is also included. To compute the asymptotic distribution, the pdf coefficients $\mathbf{h}_k$ were computed, and 10^5 samples of $\{Z_{k,r}\}_{k=1,r=1}^{K_{\max},d_{q,k}}$, with $K_{\max} = 10$, were generated according to Proposition 5.2.



(a) von Mises–Fisher

(b) Watson

Figure 14: Location parameters $\boldsymbol{\mu} \in \mathbb{S}^q$, $q \in \{1, 2\}$, considered in Figures 15–22.

The resulting histograms under scenario (i) are displayed in Figures 15–16 for $q = 1$, corresponding to V- and U-statistics, respectively, and in Figures 19–20 for $q = 2$. Under scenario (ii), the histograms are shown in Figures 17–18 ($q = 1$) and Figures 21–22 ($q = 2$).

For each set of histograms corresponding to a specific statistic and scenario, we assess the effect of rotation by performing the k-sample Anderson–Darling test on the asymptotic distributions obtained for different location parameters $\boldsymbol{\mu}$. Specifically, we draw a subsample of size 10^3 from each asymptotic distribution (one per $\boldsymbol{\mu}$) and use these subsamples to conduct the test. This allows us to evaluate whether the asymptotic distribution varies with the rotation of the alternative. The results are interesting: for odd values of m, the asymptotic distribution clearly depends on the choice of the location parameter $\boldsymbol{\mu}$, for both U- and V-statistics. However, for even values of m, the behavior of U- and V-statistics seems to differ. While for V-statistics the distribution exhibits consistent invariance to $\boldsymbol{\mu}$, for U-statistics it varies significantly in certain scenarios and values of m, indicating sensitivity to rotation even in the even-m case.

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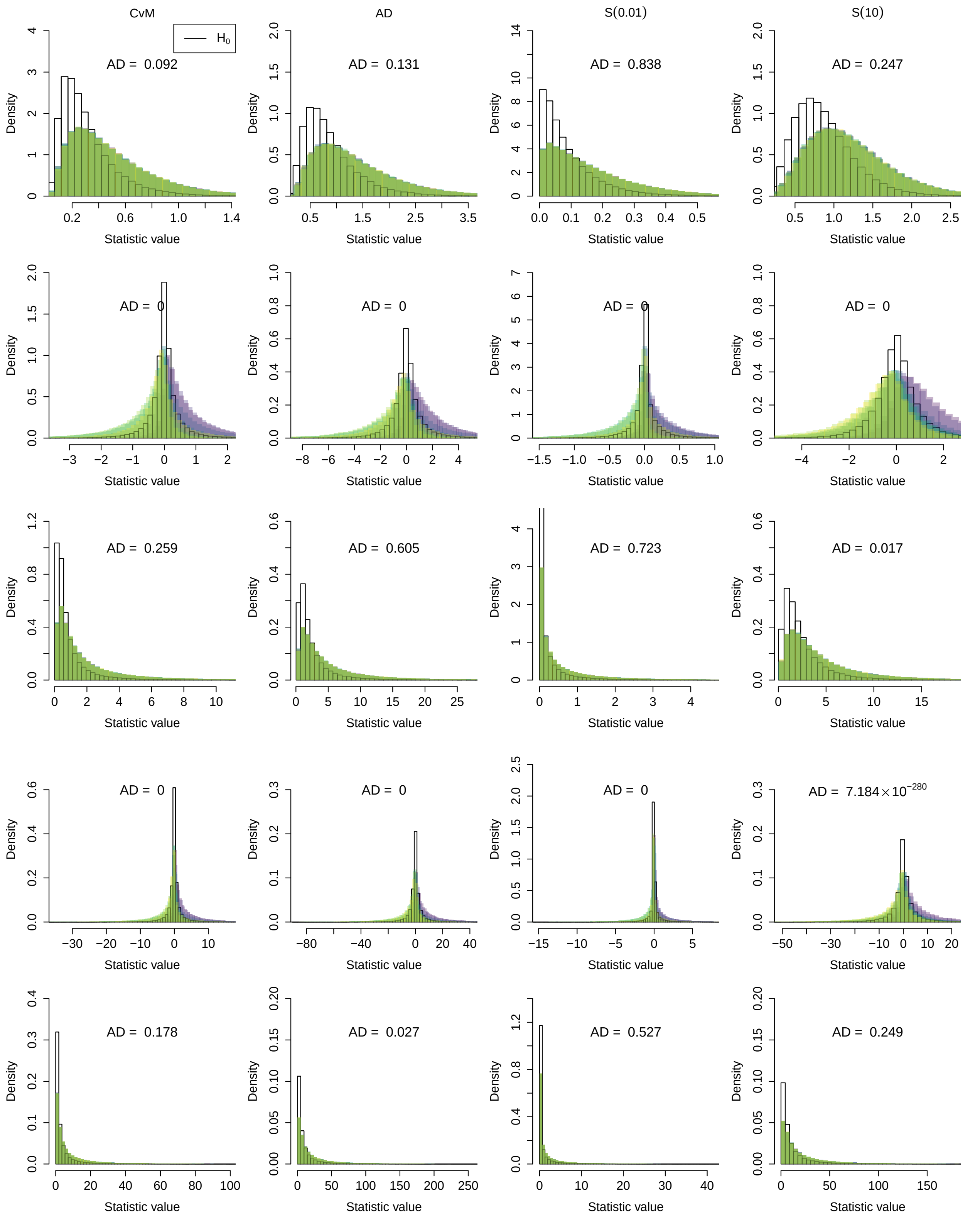

Figure 15: Histograms of the asymptotic distribution of $V_{m,w,10}$ with $m \in \{2, 3, 4, 5, 6\}$, respectively for each row, and w indicated by columns. The asymptotic distribution is shown under h_n ($q = 1$) based on vMF with $\boldsymbol{\mu}$ indicated in $(a)(i)$ and represented with varying colors. The p-value of the k-sample Anderson–Darling (AD) test for the asymptotic distributions under h_n is shown.

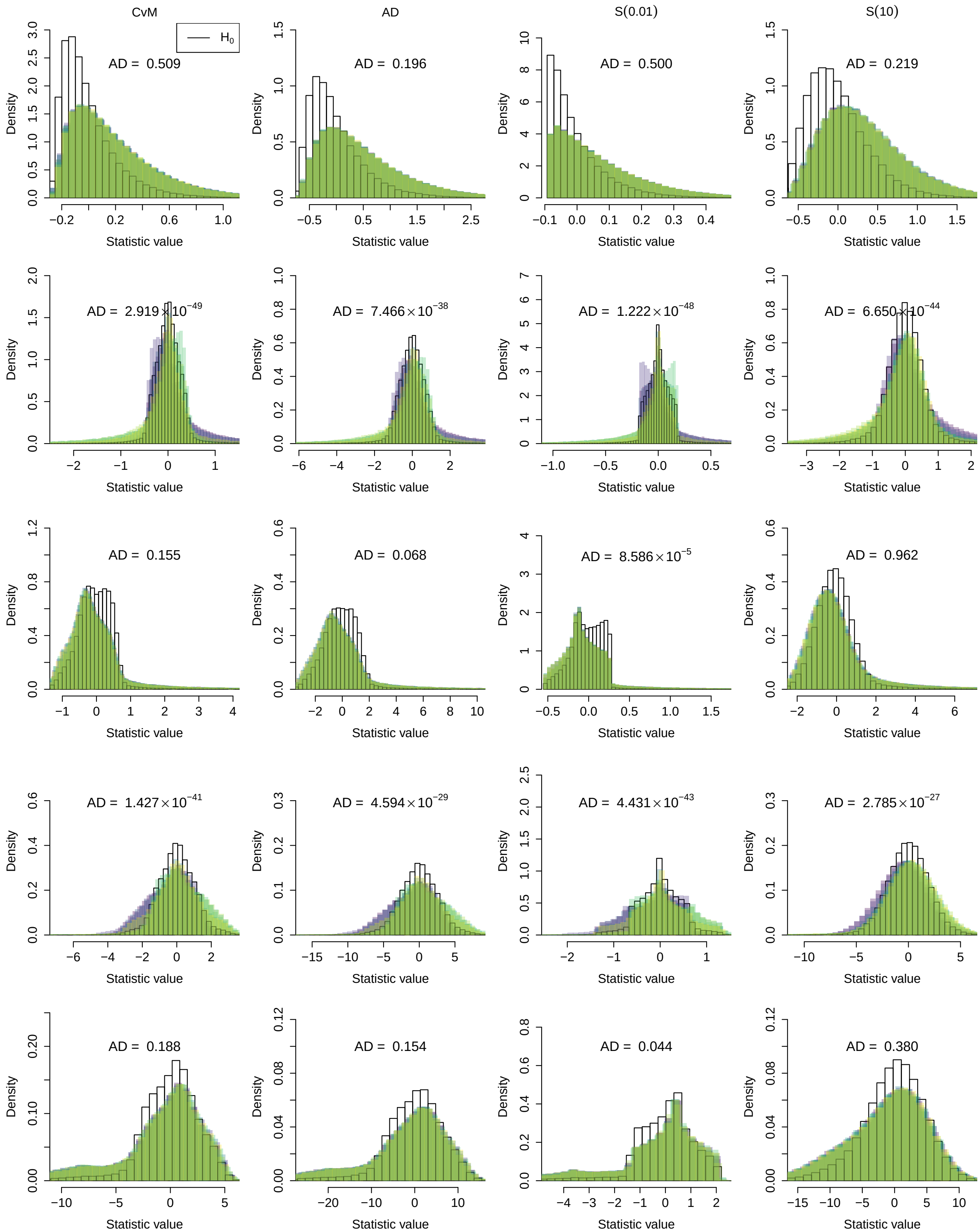

Figure 16: Histograms of the asymptotic distribution of $U_{m,w,10}$ under h_n ($q = 1$) based on vMF with $\boldsymbol{\mu}$ indicated in $(a)(i)$. The same description of Figure 15 applies.

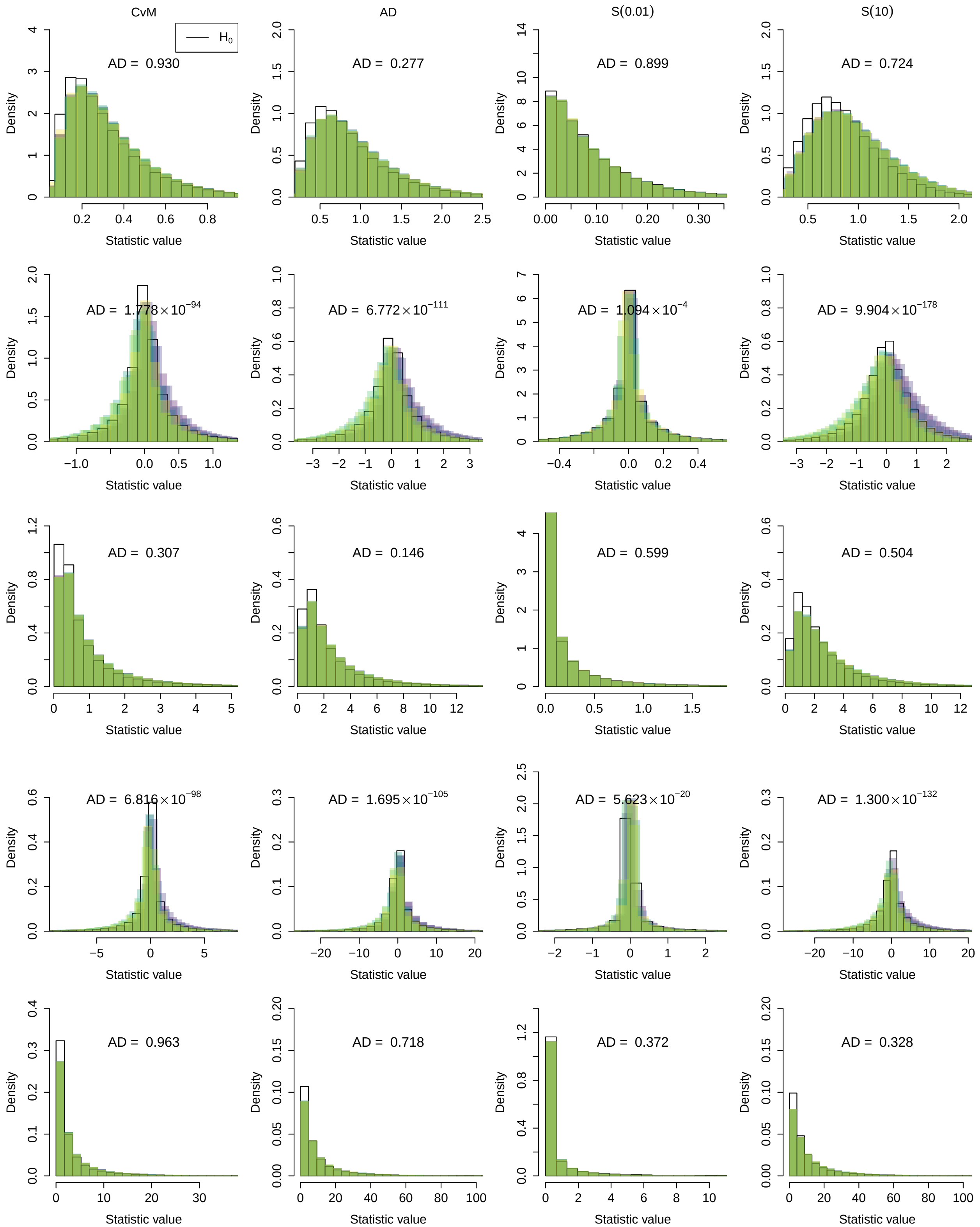

Figure 17: Histograms of the asymptotic distribution of $V_{m,w,10}$ under h_n $(q = 1)$ based on Watson with $\boldsymbol{\mu}$ indicated in $(a)(ii)$. The same description of Figure 15 applies.

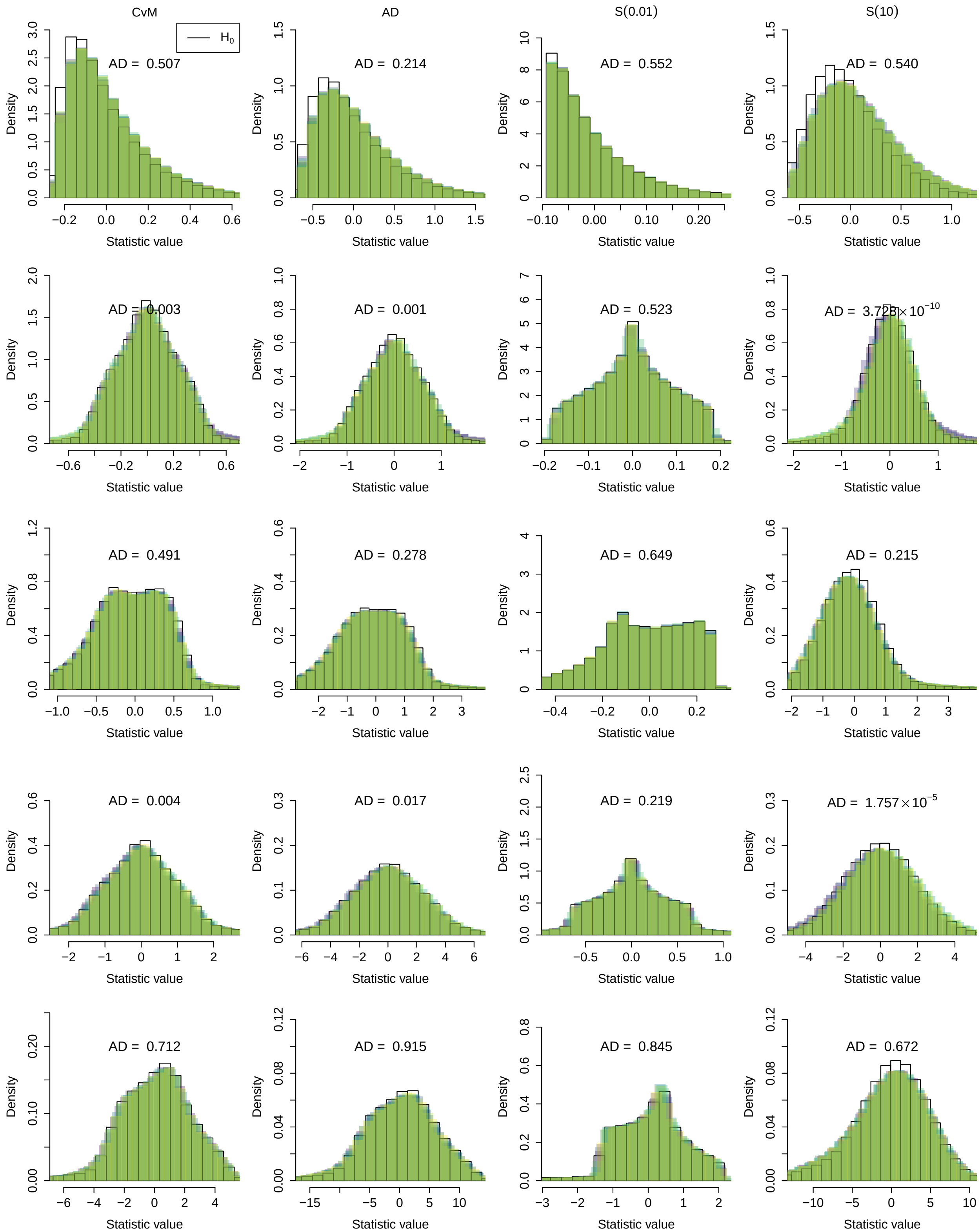

Figure 18: Histograms of the asymptotic distribution of $U_{m,w,10}$ under h_n $(q = 1)$ based on Watson with $\boldsymbol{\mu}$ indicated in $(a)(ii)$. The same description of Figure 15 applies.

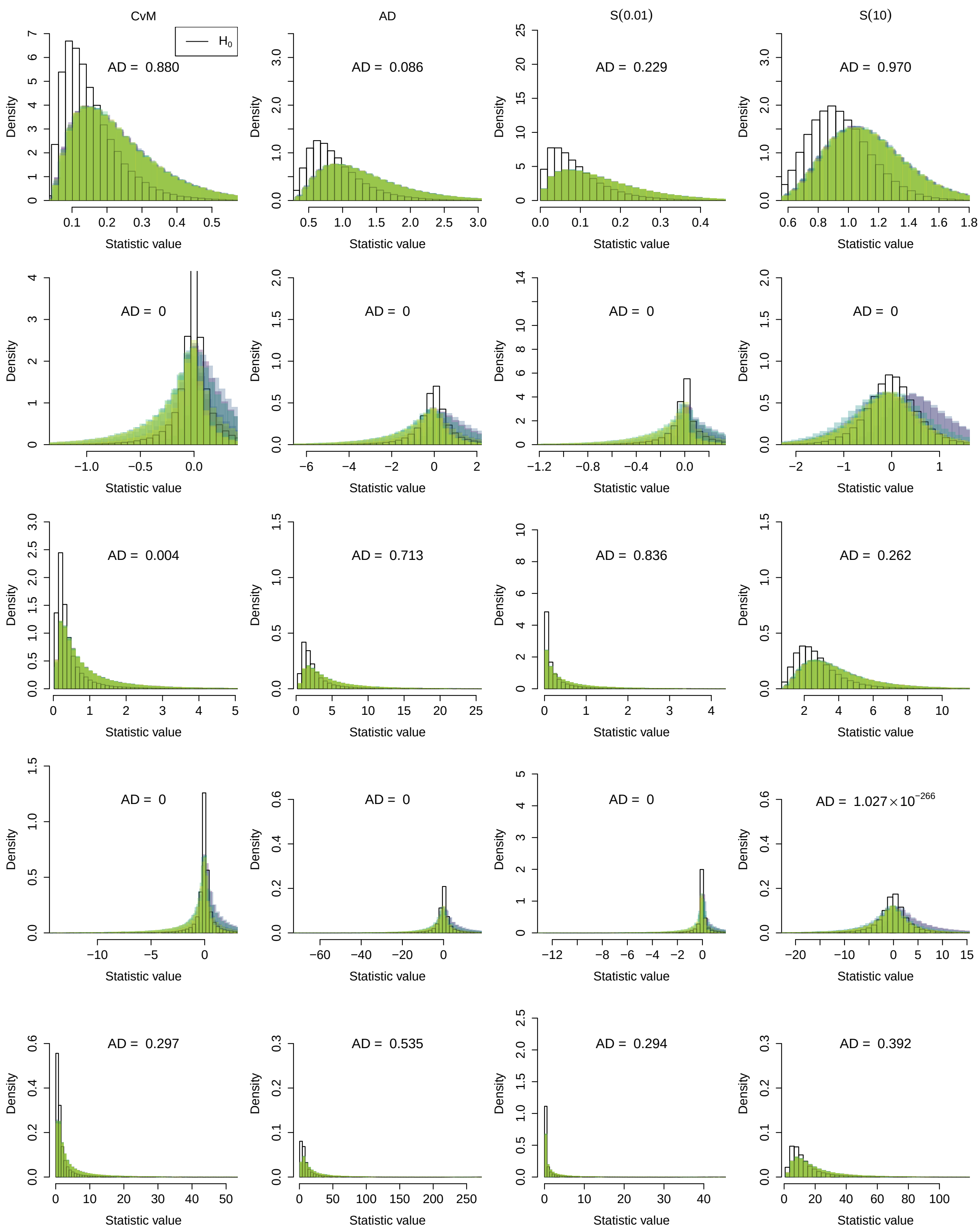

Figure 19: Histograms of the asymptotic distribution of $V_{m,w,10}$ under h_n ($q = 2$) based on vMF with $\boldsymbol{\mu}$ indicated in $(b)(i)$. The same description of Figure 15 applies.

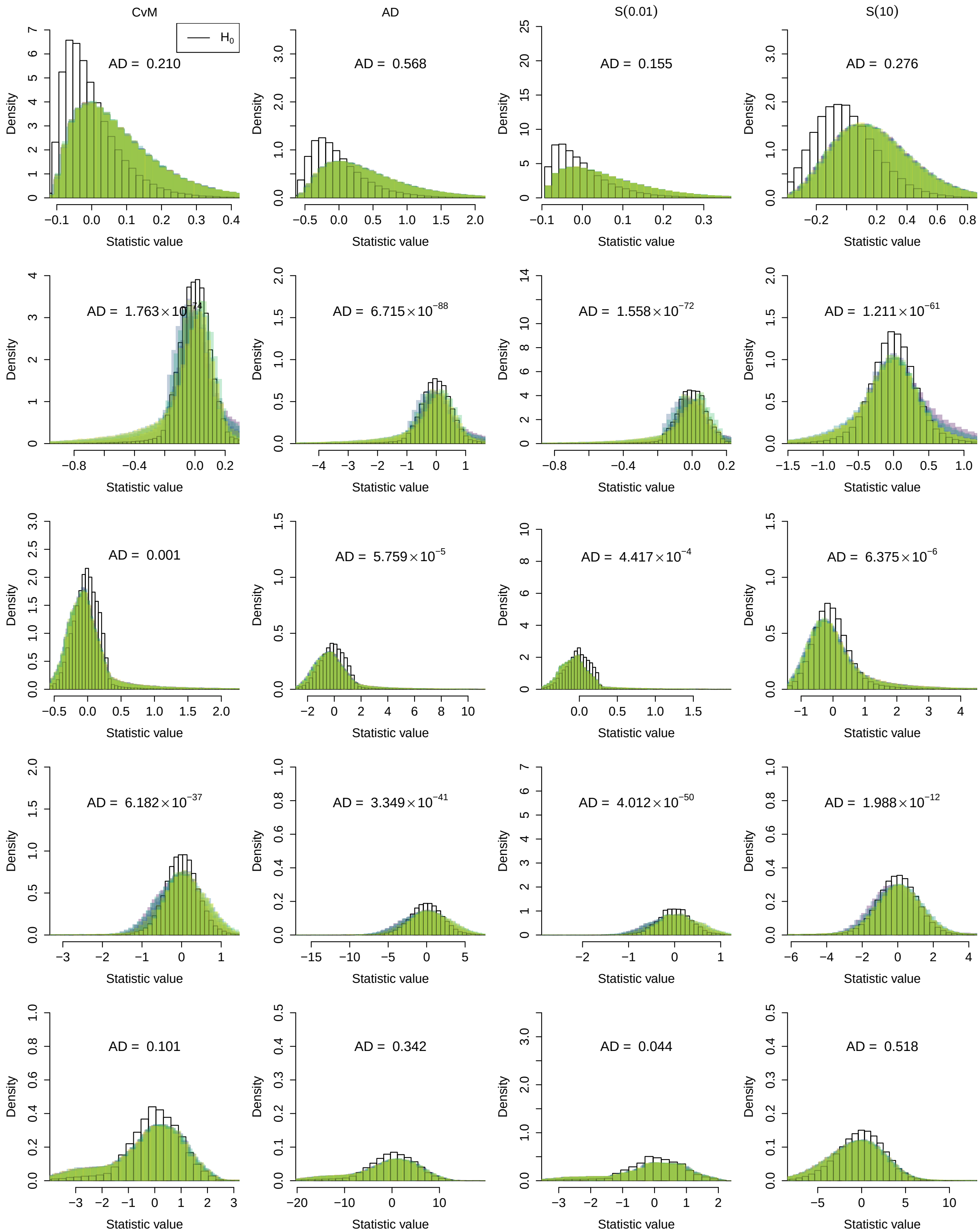

Figure 20: Histograms of the asymptotic distribution of $U_{m,w,10}$ under h_n ($q = 2$) based on vMF with $\boldsymbol{\mu}$ indicated in $(b)(i)$. The same description of Figure 15 applies.

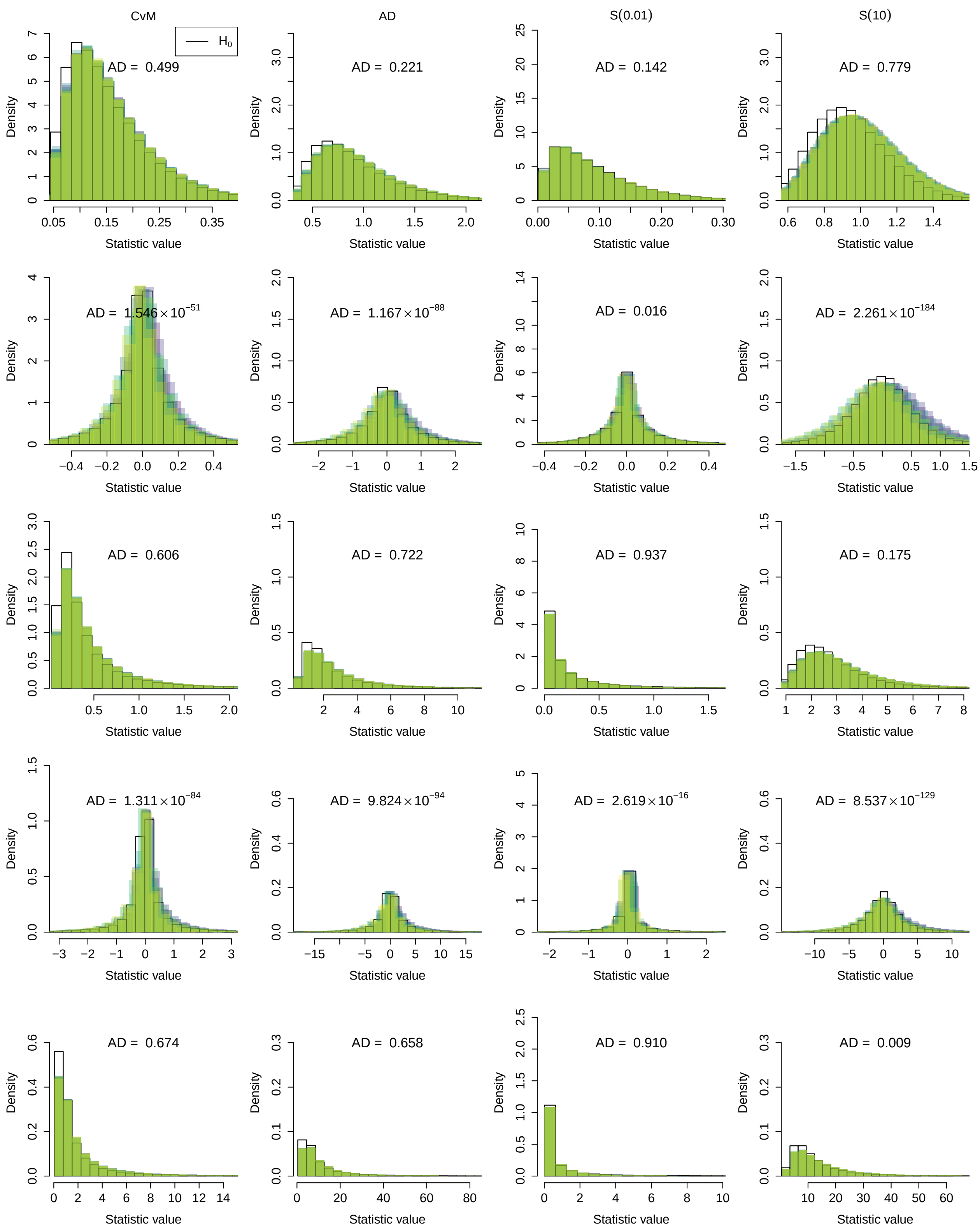

Figure 21: Histograms of the asymptotic distribution of $V_{m,w,10}$ under h_n $(q = 2)$ based on Watson with $\boldsymbol{\mu}$ indicated in $(b)(ii)$. The same description of Figure 15 applies.

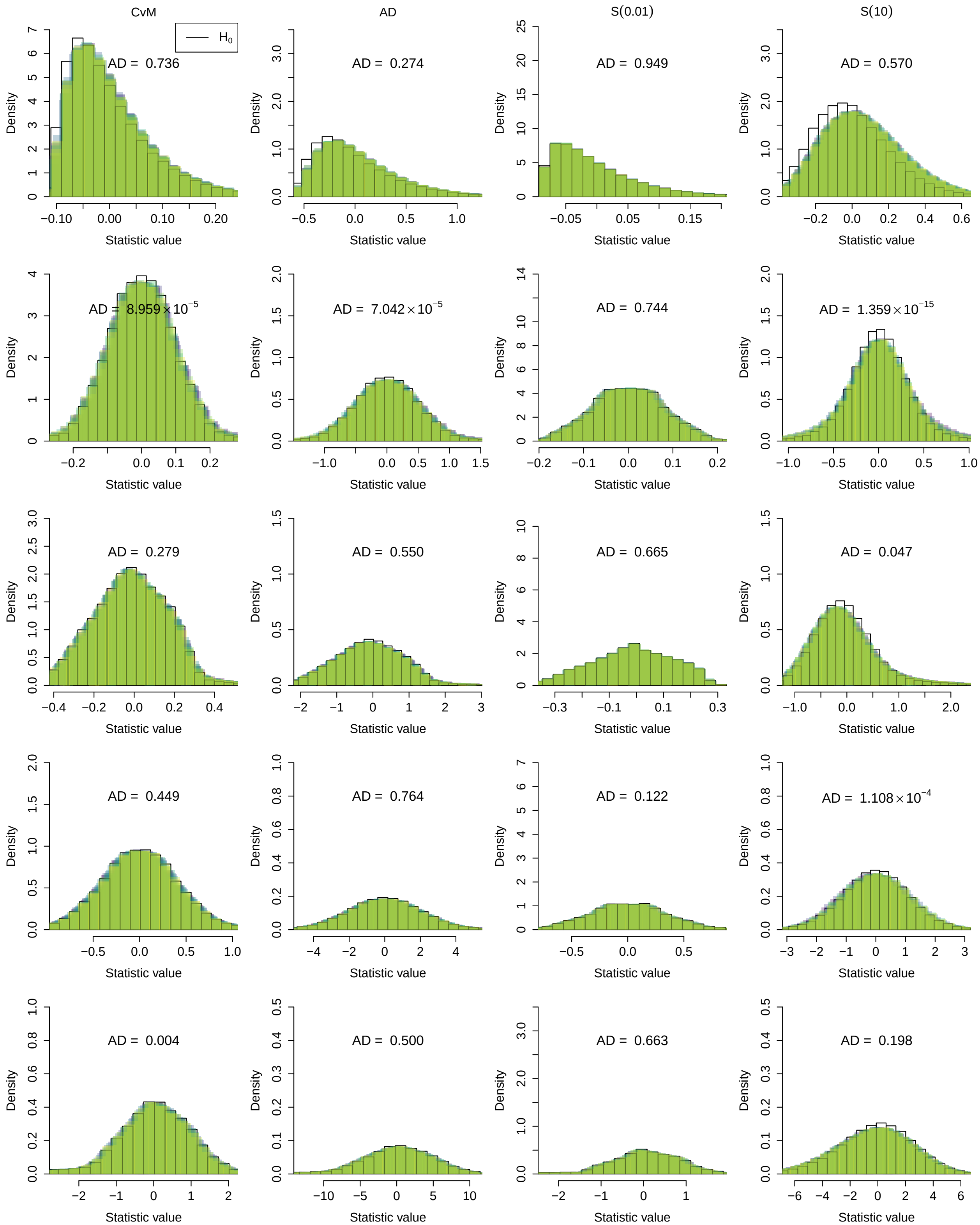

Figure 22: Histograms of the asymptotic distribution of $U_{m,w,10}$ under h_n $(q = 2)$ based on Watson with $\boldsymbol{\mu}$ indicated in $(b)(ii)$. The same description of Figure 15 applies.

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